

Review of Quantum Mechanics

Cohen-Tannoudji Ch. II & III, Preskill 2.1 & 2.3

Review of Quantum Mechanics

Note: Everyone is assumed to be familiar with grad level QM



Quick review focused on 2-level systems, Tensor Product spaces and Density Matrix formalism

State vectors (“Rays” in Preskill)

Unique quantum state \leftrightarrow unique state vector

$|\psi\rangle \in \mathcal{E}$ \leftarrow State Space

Scalar product

$$\langle \phi | \psi \rangle = \langle \psi | \phi \rangle^*$$

complex number \nearrow

(\mathcal{E} is a Hilbert Space)

Linear Operators

$$\forall |\psi\rangle \in \mathcal{E}: A|\psi\rangle = |\psi'\rangle \in \mathcal{E}$$

Projectors $P_\psi = |\psi\rangle\langle\psi|$ \leftarrow Projector on $|\psi\rangle$

$$P_{\mathcal{E}_q} = \sum_{i=1}^{g_q} |\phi_q^i\rangle\langle\phi_q^i| \leftarrow \text{projector on subspace } \mathcal{E}_q$$

\nwarrow Basis in g_q dimensional \mathcal{E}_q

Hermitian Operators $A^\dagger = A$

Adjoint $|\psi'\rangle = A|\psi\rangle \leftrightarrow \langle\psi'| = \langle\psi|A^\dagger$

Physical (measurable) quantities!

Review of Quantum Mechanics

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Projectors $P_\psi = |\psi\rangle\langle\psi|$ ← Projector on $|\psi\rangle$

$$P_{\mathcal{E}_g} = \sum_{i=1}^{g_g} |\varphi_g^i\rangle\langle\varphi_g^i|$$

← projector on subspace \mathcal{E}_g
← Basis in g_g dimensional \mathcal{E}_g

Hermitian Operators $A^\dagger = A$

Adjoint $|\psi'\rangle = A|\psi\rangle \iff \langle\psi'| = \langle\psi|A^\dagger$

Physical (measurable) quantities!

Eigenvalue Equation

$$A|\psi\rangle = \lambda|\psi\rangle$$

A Hermitian

* Eigenvalues of A are real-valued

* Eigenvectors $A|\psi\rangle = \lambda|\psi\rangle$ are orthogonal
 $A|\varphi\rangle = \mu|\varphi\rangle$ if $\lambda \neq \mu$

* Eigenvectors of A form orthonormal basis in \mathcal{E}

Commuting Observables

$$[A, B] \equiv AB - BA = 0 \implies$$

\exists orthonormal basis in \mathcal{E} of common eigenvectors of A, B

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C.S.C.O (Complete set of commuting observables)

Set A, B, C, \dots such that basis \exists in \mathcal{E} of eigenvectors $|a_m, b_m, c_m, \dots\rangle$ uniquely labeled by the set of eigenvalues a_m, b_m, c_m

Example H, L^2, L_z for the Hydrogen atom

Unitary Operators

U is unitary $\Rightarrow U^{-1} = U^\dagger \Leftrightarrow U^\dagger U = U U^\dagger = \mathbb{1}$

Scalar product invariant: $\langle\psi|\phi\rangle = \langle\psi|U^\dagger U|\phi\rangle$

$\Rightarrow U$ is a change of basis in \mathcal{E}

$U|\psi\rangle = \lambda|\psi\rangle \Rightarrow \lambda = e^{i\theta}$

eigenvcs for $\lambda \neq \lambda'$ are orthogonal

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Representation and bases

The set $\{|u_i\rangle\}$ forms a basis in \mathcal{E} if the expansion

$$|\psi\rangle = \sum_i \langle u_i | \psi \rangle |u_i\rangle \quad \text{is unique and exists} \quad \forall |\psi\rangle \in \mathcal{E}$$

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States

$$|\psi\rangle \Leftrightarrow \begin{bmatrix} \vdots \\ \langle u_i | \psi \rangle \\ \vdots \end{bmatrix}$$

Operators

$$A \Leftrightarrow \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{bmatrix}$$

Review of Quantum Mechanics

Postulates of Quantum Mechanics

- (1) At a fixed time t the state of a physical system is defined by specifying a ket $|\psi(t)\rangle$ belonging to the state space \mathcal{E} .
- (2) Every measurable physical quantity \mathcal{A} is described by an operator A acting in \mathcal{E} ; this operator is an observable.
- (3) The only possible result of a measurement of a physical quantity \mathcal{A} is one of the eigenvalues of the corresponding observable A .
- (4) (Discrete non-degenerate spectrum)
When the physical quantity \mathcal{A} is measured on a system in the normalized state $|\psi\rangle$, the probability $\mathcal{P}(a_n)$ of obtaining the non-degenerate eigenvalue a_n of the observable A is:
$$\mathcal{P}(a_n) = |\langle a_n | \psi \rangle|^2 = \langle \psi | P_n | \psi \rangle$$
where $|a_n\rangle$ is the normalized eigenvector of A associated with the eigenvalue a_n , and $P = |a_n\rangle\langle a_n|$ is the projector onto $|a_n\rangle$.

Postulates of Quantum Mechanics

- (5) If the measurement of the physical quantity \mathcal{A} on the system in state $|\psi\rangle$ gives the result a_n , then the state immediately after the measurement is the normalized projection of $|\psi\rangle$ onto $|a_n\rangle$:

$$|\psi_{\text{after}}\rangle = \frac{P_n |\psi\rangle}{\langle \psi | P_n | \psi \rangle}$$

Degenerate case: use projector onto the Subspace associated with a_n .

- (6) The time evolution of the state vector $|\psi(t)\rangle$ is governed by the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

where $H(t)$ is the observable associated with the total energy of the system.

See also Note on the **Bayesian Update Rule** for “classical” probability distributions

Tensor Products of State Spaces

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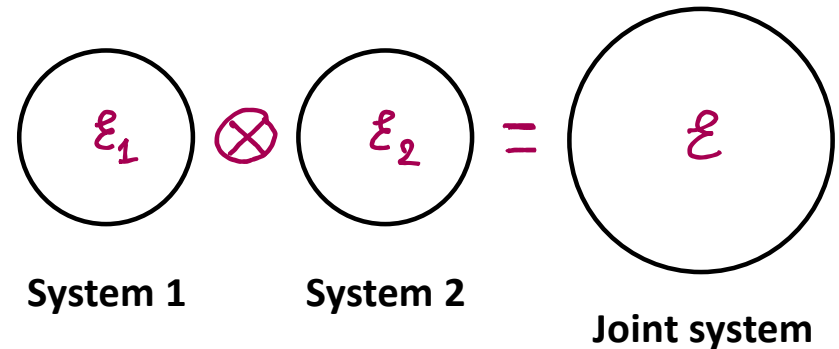
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Quantum Mechanics of systems that consist of multiple parts



Def: Let E_1, E_2 be vector spaces of dimension N_1, N_2

The vector space $E = E_1 \otimes E_2$ is called the Tensor Product of E_1 and E_2 iff

\forall pairs $|\varphi(1)\rangle \in E_1, |\chi(2)\rangle \in E_2, \exists$ vector $\in E$

such that

1. The association is linear with respect to multiplication with complex numbers

$$\lambda |\varphi(1)\rangle \otimes \mu |\chi(2)\rangle = \lambda \mu [|\varphi(1)\rangle \otimes |\chi(2)\rangle]$$

Tensor Products of State Spaces

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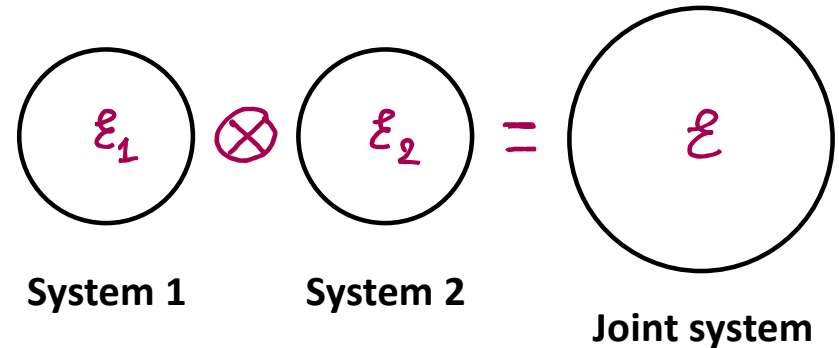
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Tensor Products of State Spaces

2. Distributive $|\varphi\rangle \otimes [a|\chi_1\rangle + b|\chi_2\rangle]$
 $= a|\varphi\rangle \otimes |\chi_1\rangle + b|\varphi\rangle \otimes |\chi_2\rangle$

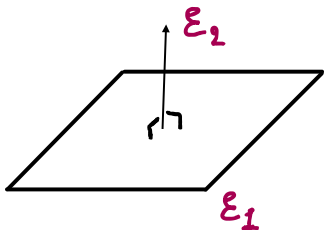
2. Bases $\{|\mu_i\rangle\}$ in \mathcal{E}_1 , $\{|\nu_e\rangle\}$ in \mathcal{E}_2

→ $\{|\mu_i\rangle \otimes |\nu_e\rangle\}$ is a basis in \mathcal{E}

Iff N_1, N_2 are finite, then $\text{Dim}(\mathcal{E}) = N_1 \times N_2$

These properties → The usual linear algebra works in \mathcal{E}

Analogy: Tensor product of 1D & 2D geometrical space



Note: $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \neq 3D$ geom. space

SP of vectors in \mathcal{E}_1 w/vectors in \mathcal{E}_2
not defined

Vectors in \mathcal{E}

Let

$$|\varphi\rangle = \sum a_i |\mu_i\rangle$$

$$|\chi\rangle = \sum b_e |\nu_e\rangle$$

Then $|\varphi\rangle \otimes |\chi\rangle = \sum_{i,e} a_i b_e |\mu_i\rangle \otimes |\nu_e\rangle$

Hugely important:

There are vectors in \mathcal{E} that are not tensor products of vectors from $\mathcal{E}_1, \mathcal{E}_2$

General vector $e \mathcal{E}$ can be written as

$$|\psi\rangle = \sum_{i,e} c_{i,e} |\mu_i\rangle \otimes |\nu_e\rangle$$

How to see? There are $N_1 \times N_2$ prob. ampl's $c_{i,e}$

These cannot all be written as $a_i \times b_e$ where the sets $\{a_i\}, \{b_e\}$ are valid probability amplitudes.

Tensor Products of State Spaces

Vectors in \mathcal{E}

Let

$$|\varphi(1)\rangle = \sum a_i |\mu_i(1)\rangle$$

$$|\chi(2)\rangle = \sum b_e |\nu_e(2)\rangle$$

Then $|\varphi(1)\rangle \otimes |\chi(2)\rangle = \sum_{i,e} a_i b_e |\mu_i(1)\rangle \otimes |\nu_e(2)\rangle$

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Example: $\mathcal{E}_1, \mathcal{E}_2$ are qubits, $N_1 = N_2 = 2$

$$|\varphi(1)\rangle = a_1 |\mu_1(1)\rangle + a_2 |\mu_2(1)\rangle$$

$$|\chi(2)\rangle = b_1 |\nu_1(2)\rangle + b_2 |\nu_2(2)\rangle$$

→ 2 real-valued variables each

In basis $\{|\mu_i(1)\rangle \otimes |\nu_e(2)\rangle\}$

Product state

$$\begin{bmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{bmatrix}$$

↑
4 real-valued variables

General state

$$\begin{bmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{bmatrix}$$

↑
6 real-valued variables

N qubits → $\begin{cases} \text{product state} \rightarrow 2N \text{ real variables} \\ \text{general state} \rightarrow 2^{N+1} - 2 \text{ real var's} \end{cases}$

Tensor Products of State Spaces

Example: $\mathcal{E}_1, \mathcal{E}_2$ are qubits, $N_1 = N_2 = 2$

$$|\varphi(1)\rangle = a_1 |u_1(1)\rangle + a_2 |u_2(1)\rangle$$

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Note: States $\in \mathcal{E}$ that are not product states are known as

Entangled States or Correlated States

Back to the Linear Algebra engine of QM

Scalar product: $(\langle \varphi'(1) | \otimes \langle \chi'(2) |) (|\varphi(1)\rangle \otimes |\chi(2)\rangle)$
 $= \langle \varphi'(1) | \varphi(1) \rangle \langle \chi'(2) | \chi(2) \rangle$

Operators: Let $A(1)$ act in $\mathcal{E}(1)$

The Extension $\tilde{A}(1)$ acting in \mathcal{E} is defined by

$$\tilde{A}(1) [|\varphi(1)\rangle \otimes |\chi(2)\rangle] = (A(1)|\varphi(1)\rangle) \otimes |\chi(2)\rangle$$

Extension $\tilde{B}(2)$ of $B(2)$ into \mathcal{E} is similar

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Tensor Product of Operators

$$[A(1) \otimes B(2)] [|\varphi(1)\rangle \otimes |\chi(2)\rangle] = [A(1)|\varphi(1)\rangle] \otimes [B(2)|\chi(2)\rangle]$$

$$\Rightarrow A(1) \otimes B(2) = \tilde{A}(1) \tilde{B}(2)$$

special case:

$$\tilde{A}(1) = A(1) \otimes \mathbb{1}(2)$$

$$\tilde{B}(2) = \mathbb{1}(1) \otimes B(2)$$

Commutator

$$[\tilde{A}(1), \tilde{B}(2)] = 0 \text{ because } [A(1), \mathbb{1}(1)] = [B(2), \mathbb{1}(2)] = 0$$

Notation: Obvious from context

$$|\varphi(1)\rangle \otimes |\chi(2)\rangle \leftrightarrow |\varphi(1)\chi(2)\rangle \leftrightarrow |\varphi(1)\rangle |\chi(2)\rangle$$

$$A(1) \otimes B(2) \leftrightarrow A(1)B(2)$$

$$\tilde{A}(1) \leftrightarrow A(1)$$

Tensor Products of State Spaces

Tensor Product of Operators

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Eigenvalue problem in \mathcal{E}

Let $A(1)|\varphi_n^i(1)\rangle = a_n |\varphi_n^i(1)\rangle, i=1, \dots, g_n \Rightarrow$

$$A(1)|\varphi_n^i(1)\chi(2)\rangle = a_n |\varphi_n^i(1)\chi(2)\rangle \quad \forall |\chi(2)\rangle \in \mathcal{E}_2$$

Can choose $|\chi(2)\rangle \in$ orthonormal basis in \mathcal{E}_2

$$\Rightarrow g_i = N_2 \text{ - fold degeneracy of } a_n \text{ in } \mathcal{E}$$

Furthermore

$$\left. \begin{aligned} A(1)|\varphi_n^i(1)\rangle &= a_n |\varphi_n^i(1)\rangle \\ B(2)|\chi_e^j(2)\rangle &= b_e |\chi_e^j(2)\rangle \end{aligned} \right\} \Rightarrow$$

$$(A(1) + B(2))|\varphi_n^i(1)\chi_e^j(2)\rangle = (a_n + b_e) |\varphi_n^i(1)\chi_e^j(2)\rangle$$

$$A(1)B(2)|\varphi_n^i(1)\chi_e^j(2)\rangle = a_n b_e |\varphi_n^i(1)\chi_e^j(2)\rangle$$

$$f(A(1), B(2))|\varphi_n^i(1)\chi_e^j(2)\rangle = f(a_n, b_e) |\varphi_n^i(1)\chi_e^j(2)\rangle$$

Postulates of QM apply in $\mathcal{E}_1, \mathcal{E}_2$ and \mathcal{E}

\Rightarrow **We are Done!**

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When the physical quantity \mathcal{A} is measured on a system in the normalized state $|\psi\rangle$, the probability $P(a_n)$ of obtaining the non-degenerate eigenvalue a_n of the observable A is:
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See also Note on the **Bayesian Update Rule** for “classical” probability distributions

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Bayes rule and the updating of probabilities

Consider two stochastic variables X and Y .
We have

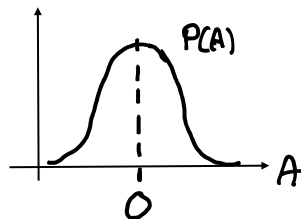
$$P(x,y) = P(x|y)P(y) = P(y|x)P(x)$$

$$P(x|y) = \frac{P(x,y)P(x)}{P(y)} \quad \text{and} \quad P(y|x) = \frac{P(x,y)P(y)}{P(x)}$$

Consider a classical particle located somewhere on the x -axis. The Bayesian interpretation holds that a probability distribution quantifies prior knowledge, in this example about the position of the particle.

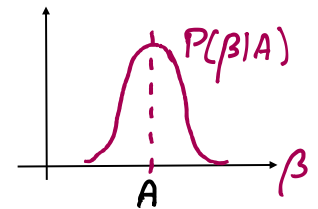
Let $P(A)$ be the probability of finding the particle at $x=A$

$$P(A) = \frac{1}{\sqrt{2\pi\sigma_A^2}} e^{-A^2/2\sigma_A^2}$$



Assume we measure the position of the particle without disturbing it. The measurement has finite resolution, quantified by the conditional probability of observing $x=\beta$ given $P(A)$

$$P(\beta|A) = \frac{1}{\sqrt{2\pi\sigma_\beta^2}} e^{-(\beta-A)^2/2\sigma_\beta^2}$$



Bayes Rule: $P(A,\beta) = P(A|\beta)P(\beta) = P(\beta|A)P(A)$ \rightarrow

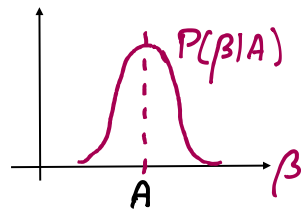
$$P(A|\beta) = \frac{P(\beta|A)P(A)}{P(\beta)} \quad \text{where} \quad P(\beta) = \int_{-\infty}^{\infty} P(\beta|A)P(A)dA$$

Note: $P(A)$, $P(\beta|A)$ & $P(\beta)$ are known beforehand and independent of the measurement outcome.

Bayes rule and the updating of probabilities

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Bayes Rule: $P(A, \beta) = P(A|\beta)P(\beta) = P(\beta|A)P(A)$ →

$$P(A|\beta) = \frac{P(\beta|A)P(A)}{P(\beta)} \quad \text{where} \quad P(\beta) = \int_{-\infty}^{\infty} P(\beta|A)P(A)dA$$

Note: $P(A)$, $P(\beta|A)$ & $P(\beta)$ are known beforehand and independent of the measurement outcome.

In our example

$$P(\beta) \propto \int_{-\infty}^{\infty} e^{-(A-\beta)^2/2\sigma_\beta^2} \times e^{-A^2/2\sigma_A^2} dA \propto e^{-\beta^2/2(\sigma_A^2 + \sigma_\beta^2)}$$

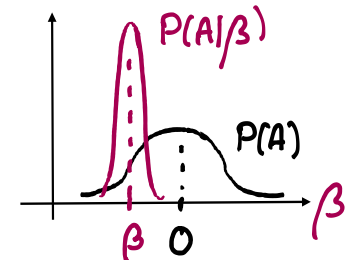
and therefore

$$P(A|\beta) = \text{const} \times e^{-(A-\beta)^2/2\sigma_\beta^2} \times e^{\beta^2/2(\sigma_A^2 + \sigma_\beta^2)}$$

Interpretation: $\sigma_\beta \ll \sigma_A$ (sharp measurement)

→ $g_2 \sim$ constant over range where $g_1 \neq 0$ →

$$P(A|\beta) \approx \frac{1}{\sqrt{2\pi\sigma_\beta^2}} e^{-(A-\beta)^2/2\sigma_\beta^2}$$



Bayes rule and the updating of probabilities

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$$P(\beta) \propto \int_{-\infty}^{\infty} e^{-(A-\beta)^2/2\sigma_\beta^2} \times e^{-A^2/2\sigma_A^2} dA \propto e^{-\beta^2/2(\sigma_A^2 + \sigma_\beta^2)}$$

and therefore

g_1

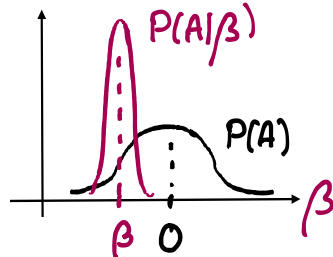
g_2

$$P(A|\beta) = \text{Const} \times e^{-(A-\beta)^2/2\sigma_\beta^2} \times e^{\beta^2/2(\sigma_A^2 + \sigma_\beta^2)}$$

Interpretation: $\sigma_\beta \ll \sigma_A$ (sharp measurement)

→ $g_2 \sim$ constant over range where $g_1 \neq 0$ →

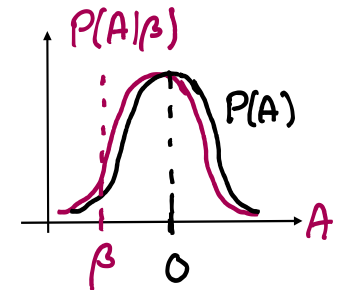
$$P(A|\beta) \approx \frac{1}{\sqrt{2\pi\sigma_\beta^2}} e^{-(A-\beta)^2/2\sigma_\beta^2}$$



Interpretation: $\sigma_\beta \gg \sigma_A$ (unsharp measurement)

→ $g_1 \approx e^{-(A-\beta)^2/2\sigma_\beta^2} \sim e^{-\beta^2/2\sigma_\beta^2}$ where $g_2 \neq 0$ →

$$P(A|\beta) \approx \frac{1}{\sqrt{2\pi\sigma_A^2}} e^{-A^2/2\sigma_A^2}$$



Note: the posterior distribution changes very little, meaning we learn little from the measurement

* The above are the Bayesian Update Rules when new information becomes available

Bayes rule and the updating of probabilities

In our example

$$P(\beta) \propto \int_{-\infty}^{\infty} e^{-(A-\beta)^2/2\sigma_\beta^2} \times e^{-A^2/2\sigma_A^2} dA \propto e^{-\beta^2/2(\sigma_A^2 + \sigma_\beta^2)}$$

and therefore

g_1

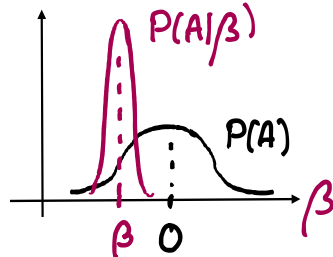
g_2

$$P(A|\beta) = \text{Const} \times e^{-(A-\beta)^2/2\sigma_\beta^2} \times e^{\beta^2/2(\sigma_A^2 + \sigma_\beta^2)}$$

Interpretation: $\sigma_\beta \ll \sigma_A$ (sharp measurement)

→ $g_2 \sim$ constant over range where $g_1 \neq 0$ →

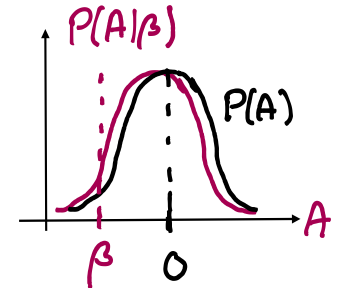
$$P(A|\beta) \approx \frac{1}{\sqrt{2\pi\sigma_\beta^2}} e^{-(A-\beta)^2/2\sigma_\beta^2}$$



Interpretation: $\sigma_\beta \gg \sigma_A$ (unsharp measurement)

→ $g_1 \approx e^{-(A-\beta)^2/2\sigma_\beta^2} \sim e^{-\beta^2/2\sigma_\beta^2}$ where $g_2 \neq 0$ →

$$P(A|\beta) \approx \frac{1}{\sqrt{2\pi\sigma_A^2}} e^{-A^2/2\sigma_A^2}$$



Note: the posterior distribution changes very little, meaning we learn little from the measurement

- * The above are the Bayesian Update Rules when new information becomes available
- * This is a strict analogy to the collapse of the quantum state following measurement