Review of Quantum Mechanics

Cohen-Tannoudji Ch. II & III, Preskill 2.1 & 2.3

Review of Quantum Mechanics

Note: Everyone is assumed to be familiar with grad level QM

Quick review focused on 2-level systems, Tensor Product spaces and Density Matrix formalism

State vectors

("Rays" in Preskill)

Unique quantum state ↔ unique state vector

$|\psi\rangle \in \mathcal{E}$ ← State Space

Scalar product

$\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^*$

complex number

($\mathcal{E}$ is a Hilbert Space)

Linear Operators

$\forall |\psi\rangle \in \mathcal{E}: \; A|\psi\rangle = |\psi'\rangle \in \mathcal{E}$

Projectors

$P_\psi = |\psi\rangle\langle\psi|$ ← Projector on $|\psi\rangle$

$P_{\mathcal{E}_q} = \sum_{i=1}^{q} |\varphi_q^i\rangle\langle\varphi_q^i|$ ← projector on subspace $\mathcal{E}_q$

Basis in q dimensional $\mathcal{E}_q$

Hermitian Operators

$A^+ = A$

Adjoint $|\psi'\rangle = A|\psi\rangle \leftrightarrow \langle\psi'| = \langle\psi|A^+$

Physical (measurable) quantities !

Review of Quantum Mechanics

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Physical (measurable) quantities!

Eigenvalue Equation

$A|\psi\rangle = \lambda|\psi\rangle$

A Hermitian

* Eigenvalues of A are real-valued
* Eigenvectors $A|\psi\rangle = \lambda|\psi\rangle$, $A|\varphi\rangle = \mu|\varphi\rangle$ are orthogonal if $\lambda \neq \mu$
* Eigenvectors of A form orthonormal basis in $\mathcal{E}$

Commuting Observables

$[A,B] \equiv AB - BA = 0$ ⇒

$\exists$ orthonormal basis in $\mathcal{E}$ of common eigenvectors of A, B

Review of Quantum Mechanics

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C.S.C.O (Complete set of commuting observables)

Set $A, B, C \ldots$ such that basis $\exists$ in $\mathcal{E}$ of eigenvectors $|a_m, b_m, c_m \ldots\rangle$ uniquely labeled by the set of eigenvalues a_m, b_m, c_m

Example H, L^2, L_z for the Hydrogen atom

Unitary Operators

U is unitary $\Rightarrow$ $U^{-1} = U^\dagger \Leftrightarrow U^\dagger U = UU^\dagger = \mathbb{1}$

Scalar product invariant: $\langle\psi|\varphi\rangle = \langle\psi|U^\dagger U|\varphi\rangle$

$\Rightarrow$ U is a change of basis in $\mathcal{E}$

$U|v\rangle = \lambda|v\rangle \Rightarrow \lambda = e^{i\theta}$

eigenvecs for $\lambda \neq \lambda'$ are orthogonal

Review of Quantum Mechanics

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Representation and bases

The set $\{|u_i\rangle\}$ forms a basis in $\mathcal{E}$ if the expansion

$$|\psi\rangle = \sum_i \langle u_i|\psi\rangle |u_i\rangle$$

is unique and exists $\forall\, |\psi\rangle \in \mathcal{E}$

States $|\psi\rangle \Leftrightarrow \begin{bmatrix} \vdots \\ \langle u_i|\psi\rangle \\ \vdots \end{bmatrix}$

Operators $A \Leftrightarrow \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$

Review of Quantum Mechanics

Postulates of Quantum Mechanics

(1) At a fixed time t the state of a physical system is defined by specifying a ket $|\psi(t)\rangle$ belonging to the state space $\mathcal{E}$.

(2) Every measurable physical quantity $\mathcal{A}$ is described by an operator A acting in $\mathcal{E}$; this operator is an observable.

(3) The only possible result of a measurement of A physical quantity $\mathcal{A}$ is one of the eigenvalues of the corresponding observable A.

(4) (Discrete non-degenerate spectrum) When the physical quantity $\mathcal{A}$ is measured on A system in the normalized state $|\psi\rangle$, the probability $\mathcal{P}(a_n)$ of obtaining the non-degenerate eigenvalue a_n of the observable A is:

$$\mathcal{P}(a_n) = |\langle a_n|\psi\rangle|^2 = \langle\psi|P_n|\psi\rangle$$

where $|a_n\rangle$ is the normalized eigenvector of A associated with the eigenvalue a_n, and $P = |a_n\rangle\langle a_n|$ is the projector onto $|a_n\rangle$.

Postulates of Quantum Mechanics

(5) If the measurement of the physical quantity $\mathcal{A}$ on the system in state $|\psi\rangle$ gives the result a_n, then the state immediately after the measurement is the normalized projection of $|\psi\rangle$ onto $|a_n\rangle$:

$$|\psi_{after}\rangle = \frac{P_n|\psi\rangle}{\langle\psi|P_n|\psi\rangle}$$

Degenerate case: use projector onto the Subspace associated with a_n.

(6) The time evolution of the state vector $|\psi(t)\rangle$ Is governed by the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

where $H(t)$ is the observable associated with the total energy of the system.

See also Note on the Bayesian Update Rule for "classical" probability distributions

Tensor Products of State Spaces

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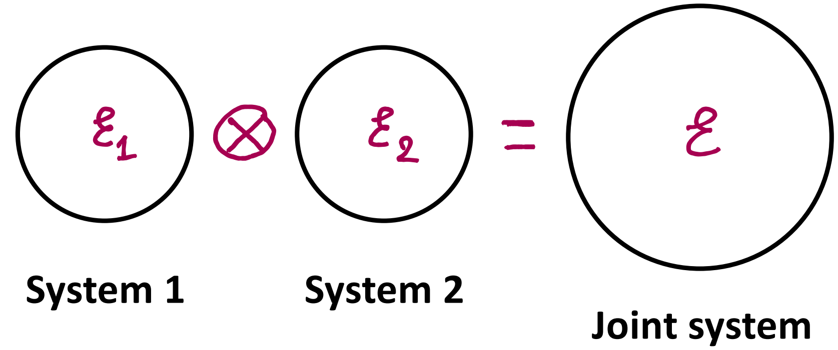
Quantum Mechanics of systems that consist of multiple parts



Def: Let $\mathcal{E}_1, \mathcal{E}_2$ be vector spaces of dimension N_1, N_2

The vector space $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ is called the <u>Tensor Product</u> of $\mathcal{E}_1$ and $\mathcal{E}_2$ iff

$\forall$ pairs $|\varphi(1)\rangle \in \mathcal{E}_1, |\chi(2)\rangle \in \mathcal{E}_2$, $\exists$ vector $\in \mathcal{E}$

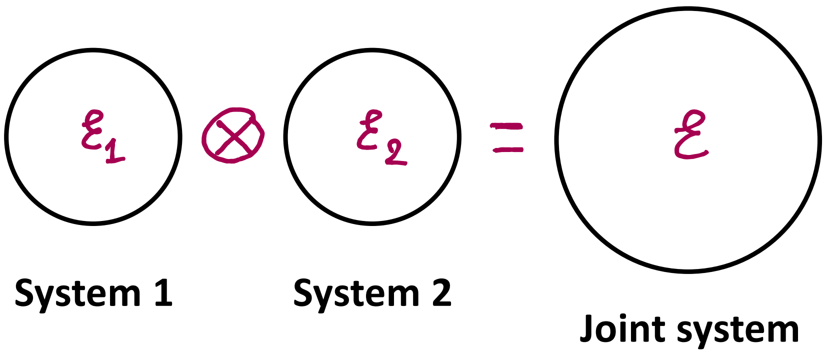
such that

1. The association is linear with respect to multiplication with complex numbers

$$\lambda|\varphi(1)\rangle \otimes \mu|\chi(2)\rangle = \lambda\mu\left[|\varphi(1)\rangle \otimes |\chi(2)\rangle\right]$$

Tensor Products of State Spaces

Quantum Mechanics of systems that consist of multiple parts



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2. Distributive $|\varphi(1)\rangle \otimes \left[a|\chi_1(2)\rangle + b|\chi_2(2)\rangle\right]$
$= a|\varphi(1)\rangle \otimes |\chi_1(2)\rangle + b|\varphi(1)\rangle \otimes |\chi_2(2)\rangle$

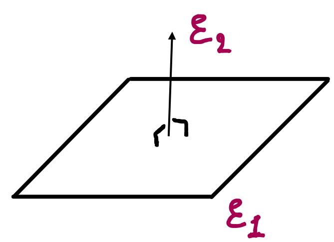
3. Bases $\{|u_i(1)\rangle\}$ in $\mathcal{E}_1$, $\{|v_l(2)\rangle\}$ in $\mathcal{E}_2$

$\Rightarrow$ $\{|u_i(1)\rangle \otimes |v_l(2)\rangle\}$ is a basis in $\mathcal{E}$

Iff N_1, N_2 are finite, then $Dim(\mathcal{E}) = N_1 \times N_2$

These properties $\Rightarrow$ The usual linear algebra works in $\mathcal{E}$

<u>Analogy</u>: Tensor product of 1D & 2D geometrical space



<u>Note</u>: $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \neq$ 3D geom. space

SP of vectors in $\mathcal{E}_1$ w/vectors in $\mathcal{E}_2$

<u>not defined</u>

Tensor Products of State Spaces

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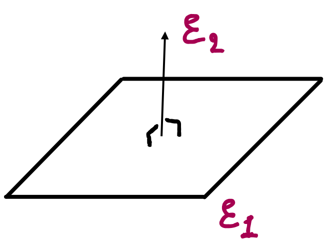

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SP of vectors in $\mathcal{E}_1$ w/vectors in $\mathcal{E}_2$ not defined

Vectors in $\mathcal{E}$

Let $|\varphi(1)\rangle = \sum a_i |u_i(1)\rangle$
$|\chi(2)\rangle = \sum b_\ell |v_\ell(2)\rangle$

Then $|\varphi(1)\rangle \otimes |\chi(2)\rangle = \sum_{i,\ell} a_i b_\ell |u_i(1)\rangle \otimes |v_\ell(2)\rangle$

Hugely important:

There are vectors in $\mathcal{E}$ that are not tensor products of vectors from $\mathcal{E}_1, \mathcal{E}_2$

General vector $\in \mathcal{E}$ can be written as

$$|\psi\rangle = \sum_{i,\ell} c_{i\ell} |u_i(1)\rangle \otimes |v_\ell(2)\rangle$$

How to see? There are $N_1 \times N_2$ prob. ampl's $c_{i\ell}$

These cannot all be written as $a_i \times b_\ell$ where the sets $\{a_i\}, \{b_\ell\}$ are valid probability amplitudes.

Tensor Products of State Spaces

Vectors in $\mathcal{E}$ Let

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Example: $\mathcal{E}_1, \mathcal{E}_2$ are qubits, $N_1 = N_2 = 2$

$$|\varphi(1)\rangle = a_1 |u_1(1)\rangle + a_2 |u_2(1)\rangle$$
$$|\chi(2)\rangle = b_1 |v_1(2)\rangle + b_2 |v_2(2)\rangle$$

$\Rightarrow$ 2 real-valued variables each

In basis $\{|u_i(1)\rangle \otimes |v_\ell(2)\rangle\}$

Product state $\begin{bmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{bmatrix}$ $\uparrow$ 4 real-valued variables

General state $\begin{bmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{bmatrix}$ $\uparrow$ 6 real-valued variables

N qubits $\Rightarrow$
- product state $\rightarrow$ $2N$ real variables
- general state $\rightarrow$ $2^{N+1} - 2$ real var's

Tensor Products of State Spaces

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N qubits ⇒
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Note: States $\in \mathcal{E}$ that are not product states are known as

Entangled States or Correlated States

End 09-20-2023

Tensor Products of State Spaces

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N qubits ➡ $\begin{cases} \text{product state} \rightarrow 2N \text{ real variables} \\ \text{general state} \rightarrow 2^{N+1} - 2 \text{ real var's} \end{cases}$

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Entangled States or Correlated States

Begin 09-25-2023

Back to the Linear Algebra engine of QM

Scalar product: $\left(\langle \varphi'(1)| \otimes \langle \chi'(2)| \right) \left(|\varphi(1)\rangle \otimes |\chi(2)\rangle \right)$

$= \langle \varphi'(1)|\varphi(1)\rangle \langle \chi'(2)|\chi(2)\rangle$

Operators: Let $A(1)$ act in $\mathcal{E}(1)$

The Extension $\tilde{A}(1)$ acting in $\mathcal{E}$ is defined by

$\tilde{A}(1) \left[|\varphi(1)\rangle \otimes |\chi(2)\rangle \right] = \left(A(1) |\varphi(1)\rangle \right) \otimes |\chi(2)\rangle$

Extension $\tilde{B}(2)$ of $B(2)$ into $\mathcal{E}$ is similar

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Tensor Product of Operators

$$[A(1)\otimes B(2)][|\varphi(1)\rangle \otimes |\chi(2)\rangle] = [A(1)|\varphi(1)\rangle] \otimes [B(2)|\chi(2)\rangle]$$

$$\Rightarrow \quad A(1)\otimes B(2) = \tilde{A}(1)\tilde{B}(2)$$

special case: $\tilde{A}(1) = A(1)\otimes \mathbb{1}(2)$

$\tilde{B}(2) = \mathbb{1}(1)\otimes B(2)$

Commutator

$[\tilde{A}(1), \tilde{B}(2)] \equiv 0$ because $[A(1), \mathbb{1}(1)] = [B(2), \mathbb{1}(2)] = 0$

Notation: Obvious from context

$$|\varphi(1)\rangle \otimes |\chi(2)\rangle \leftrightarrow |\varphi(1)\rangle|\chi(2)\rangle \leftrightarrow |\varphi(1)\chi(2)\rangle$$

$$A(1)\otimes B(2) \leftrightarrow A(1)B(2)$$

$$\tilde{A}(1) \leftrightarrow A(1)$$

Tensor Products of State Spaces

Tensor Product of Operators

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$$A(1)\otimes B(2) \leftrightarrow A(1)B(2)$$

$$\tilde{A}(1) \leftrightarrow A(1)$$

Eigenvalue problem in $\mathcal{E}$

Let $A(1)|\varphi_n^i(1)\rangle = a_n|\varphi_n^i(1)\rangle, \quad i=1,\ldots,g_n \Rightarrow$

$$A(1)|\varphi_n^i(1)\chi(2)\rangle = a_n|\varphi_n^i(1)\chi(2)\rangle \quad \forall|\chi(2)\rangle\in\mathcal{E}_2$$

Can choose $|\chi(2)\rangle\in$ orthonormal basis in $\mathcal{E}_2$

$\Rightarrow$ $g_i\times N_2$ - fold degeneracy of a_n in $\mathcal{E}$

Furthermore

$$\left.\begin{aligned} A(1)|\varphi_n^i(1)\rangle &= a_n|\varphi_n^i(1)\rangle \\ B(2)|\chi_\ell^j(2)\rangle &= b_\ell|\chi_\ell^j(2)\rangle \end{aligned}\right\} \Rightarrow$$

$$(A(1)+B(2))|\varphi_n^i(1)\chi_\ell^j(2)\rangle = (a_n+b_\ell)|\varphi_n^i(1)\chi_\ell^j(2)\rangle$$

$$A(1)B(2)|\varphi_n^i(1)\chi_\ell^j(2)\rangle = a_n b_\ell|\varphi_n^i(1)\chi_\ell^j(2)\rangle$$

$$f(A(1),B(2))|\varphi_n^i(1)\chi_\ell^j(2)\rangle = f(a_n,b_\ell)|\varphi_n^i(1)\chi_\ell^j(2)\rangle$$

Postulates of QM apply in $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}$

$\Rightarrow$ **We are Done!**

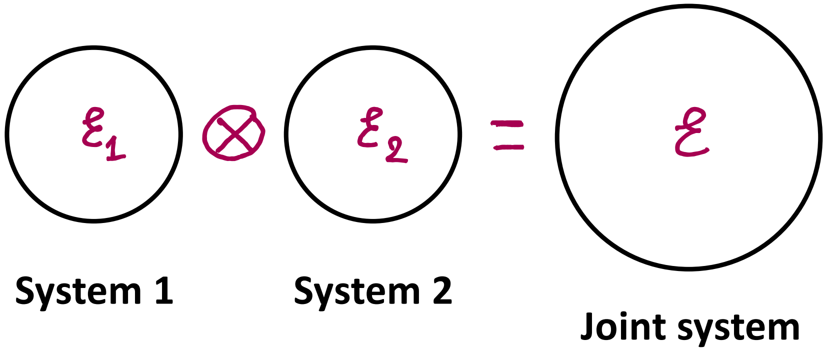
Measurement on One Part of a System

Cohen-Tannoudji Ch. III, Complement D_{III}

Begin 09-27-2023

Measurement on One Part of a System

Quantum Measurement on Bipartite Systems



Consider the following:

Bipartite System

$\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$

$\tilde{A}(1) = A(1) \otimes \mathbb{1}(2)$

Observable on System 1

Possible outcomes when measuring $\tilde{A}(1)$?

$$\boxed{\{\text{Eigenvalues of } \tilde{A}(1)\} = \{\text{Eigenvalues of } A(1)\}}$$

$\tilde{g}_n = g_n \times N_2$ $\qquad$ g_n

$\Rightarrow$ Same possible outcomes a_n indep of $|\psi\rangle$

Degeneracy in $\mathcal{E}$ increases by a factor N_2

Projector: $P_n(1) = \sum_{i=1}^{g_n} |a_n^i(1)\rangle\langle a_n^i(1)|$

for eigenvalue a_n

Using the recipe to extend an operator into $\mathcal{E}$

$$\tilde{P}_n(1) = P_n(1) \otimes \mathbb{1}(2)$$
$$= \sum_{i=1}^{g_k} \sum_{k} |a_n^i(1) v_k(2)\rangle \langle a_n^i(1)\, v_k(2)|$$

Probability of outcome a_n, $|\psi\rangle$ general state $\in \mathcal{E}$

$$p(a_n) = \langle\psi|\tilde{P}_n(1)|\psi\rangle$$
$$= \sum_{i=1}^{g_n} \sum_{k} \langle\psi|a_n^i(1) v_k(2)\rangle\langle a_n(1) v_k(2)|\psi\rangle$$

Posterior state $\quad |\psi'\rangle = \dfrac{1}{\sqrt{p(a_n)}} \tilde{P}_n(1)|\psi\rangle$

Measurement on One Part of a System

Some Observations:

1. Basis $|w_k(2)\rangle$ arbitrary, no phys. significance

2. Product States Let $|\psi\rangle = |\varphi(1)\rangle \otimes |\chi(2)\rangle$

   If we measure $A(1)$ and observe $|a_n(1)\rangle$ then

   $$|\psi'\rangle \propto P_n(1)|\varphi(1)\rangle \otimes \mathbb{1}(2)|\chi(2)\rangle \propto |\varphi'(1)\rangle \otimes |\chi(2)\rangle$$

   still a product state

3. Entangled States

   Consider a pair of states where n and i labels the eigenvalues and degeneracies within the subspace g_n

   $$|\varphi(1)\rangle = \sum_n \sum_{i=1}^{g_n} a_{ni}|u_{ni}(1)\rangle, \quad |\chi(2)\rangle = \sum_k b_k |\chi_k(2)\rangle$$

   The corresponding product state is of the form

   $$|\psi\rangle = \sum_n \sum_{i=1}^{g_n} \sum_k a_{ni} b_k |u_{ni}(1)\rangle|\chi_k(2)\rangle$$

   By comparison, the most general state in $\mathcal{E}$ has the form

   $$|\psi\rangle = \sum_n \sum_{i=1}^{g_n} \sum_k c_{nik}|u_{ni}(2)\rangle|\chi_k(2)\rangle$$

   If the c_{nik} are all products of the type $a_{ni}b_k$ then $|\psi\rangle$ is a product state. Otherwise, $|\psi\rangle$ is entangled.

Some Observations: (Continued)

3. Entangled States

   If we measure $A(1)$ and observe the outcome a_N then the posterior state is

   $$|\psi'\rangle \propto \left[P_N(1)\otimes \mathbb{1}(2)\right]|\psi\rangle \propto \sum_{i=1}^{g_N}\sum_k c_{Nik}\left[|u_{Ni}(1)\rangle \otimes |\chi_k(2)\rangle\right]$$

   Now, if $g_N = 1$ then the state $|u_N(1)\rangle$ occurs exactly once in the sum above, and therefore

   $$|\psi'\rangle \propto |u_N(1)\rangle \otimes \sum_k |\chi_k(2)\rangle \propto \left[|u_N(1)\rangle \otimes |\chi(2)\rangle\right]$$

   Conceptually, once the measurement tells us that system 1 is in the exact state $|u_N(1)\rangle$, then it factors out in the global state.

   The case $g_N > 1$ is more subtle. Once we measure a_N, we know system 1 resides in the degenerate subspace associated with the outcome a_N. Repeat measurements do not generate further information about which of the exact $|u_{Ni}(1)\rangle$ our system is in. Thus, the measurement removes some, but not all of the entanglement present in $|\psi\rangle$. To completely factorize the state we would need to measure a C.S.C.O. This will identify not only the degenerate subspace but also the specific state vector $|u_{Ni}(1)\rangle$.
   See Cohen-Tannoudji Chapter III, Complement D_{III}

Measurement on One Part of a System

Some Observations: (Continued)

3. Entangled States

If we measure $A(1)$ and observe the outcome a_N then the posterior state is

$$|\psi'\rangle \propto \left[P_N(1)\otimes \mathbb{1}(2)\right]|\psi\rangle \propto \sum_{i=1}^{g_N}\sum_{k} c_{Ni;k}\left[|u_{Ni}(1)\rangle\otimes|\chi_k(2)\rangle\right]$$

Now, if $g_N = 1$ then the state $|u_N(1)\rangle$ occurs exactly once in the sum above, and therefore

$$|\psi'\rangle \propto |u_N(1)\rangle \otimes \sum_k |\chi_k(2)\rangle \propto \left[|u_N(1)\rangle \otimes |\chi(2)\rangle\right]$$

Conceptually, once the measurement tells us that system 1 is in the exact state $|u_N(1)\rangle$, then it factors out in the global state.

The case $g_N > 1$ is more subtle. Once we measure a_N, we know system 1 resides in the degenerate subspace associated with the outcome a_N. Repeat measurements do not generate further information about which of the exact $|u_{Ni}(1)\rangle$ our system is in. Thus, the measurement removes some, but not all of the entanglement present in $|\psi\rangle$. To completely factorize the state we would need to measure a C.S.C.O. This will identify not only the degenerate subspace but also the specific state vector $|u_{Ni}(1)\rangle$.
See Cohen-Tannoudji Chapter III, Complement D_{III}

Physical Interpretation of T.P. States

From (2) above, measuring $A(1)$, $B(2)$

$$P(a_n, b_k) = \langle\varphi(1)|P_n(1)|\varphi(1)\rangle\langle\chi(2)|P_k(2)|\chi(2)\rangle$$

⇨ Outcomes a_n, b_n are **Independent Random Var's**

Uncorrelated

Physical Interpretation of Entangled States

From (3) above, measuring $A(1)$, $B(2)$

Global $|\psi\rangle$ cannot be written as $|\varphi(1)\rangle\otimes|\chi(2)\rangle$

$$P(a_n, b_k) = \langle\psi|P_n(1)P_k(2)|\psi\rangle$$

In general, a_n & b_k will be correlated random variables

Conclusion: We cannot assign state vectors to the individual subsystems !

Measurement on One Part of a System

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Even though we cannot assign $|\varphi(1)\rangle, |\chi(2)\rangle$, it is still possible to have a local description of each subsystem on its own. It must be consistent with tensor product states, yet it must reduce the information that is locally available when the global $|\psi\rangle$ is entangled

Density Matrix Formalism

Definition: A system for which we know only the probabilities p_k of finding the system in state $|\psi_k\rangle$ is said to be in a statistical mixture of states. Shorthand: mixed state.

Shorthand for non-mixed state: pure state

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Definition: Density Operator for pure states

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)|$$

Definition: Density Matrix

$$|\psi(t)\rangle = \sum_n c_n(t)|u_n\rangle$$

$$\rho_{pn}(t) = \langle u_p|\rho(t)|u_n\rangle = c_p(t)\, c_n^*(t)$$

Definition: Density Operator for mixed states

$$\rho(t) = \sum_{\vec{k}} p_k \rho_k(t), \quad \rho_k = |\psi_k(t)\rangle\langle\psi_k(t)|$$

Note: A pure state is just a mixed state for which one $p_k = 1$ and the rest are zero.

The terms Density Operator and Density Matrix are used interchangeably

Measurement on One Part of a System

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Let A be an observable w/eigenvalues a_n

Let P_n be the projector on the eigen-subspace of a_n

For a pure state, $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$, we have

(∗) $Tr\,\rho(t) = \sum_n \rho_{nn}(t) = \sum_n |c_n|^2 = 1$

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$= Tr[\rho(t)A]$ ($|u_p\rangle$ basis in $\mathcal{H}$)

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$$\mathcal{P}(a_n) = \langle\psi(t)|P_n|\psi(t)\rangle = Tr[\rho(t)P_n]$$

(∗) $\dot{\rho}(t) = |\dot{\psi}(t)\rangle\langle\psi(t)| + |\psi(t)\rangle\langle\dot{\psi}(t)|$

$= \frac{1}{i\hbar}H|\psi(t)\rangle\langle\psi(t)| - \frac{1}{i\hbar}|\psi(t)\rangle\langle\psi(t)|H$

$= \frac{1}{i\hbar}[H,\rho]$

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Density Operator formalism is general !

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Density Operator formalism is general !

Important properties of the Density Operator

(1) ρ is Hermitian, $\rho^+ = \rho$ ⇨ ρ is an observable

⇨ $\exists$ basis in which ρ is diagonal

In this basis a pure state has one diagonal element $= 1$, the rest $= 0$

(2) Test for purity.

Pure: $\rho^2 = \rho$ ⇨ $Tr\,\rho^2 = 1$

Mixed: $\rho^2 \neq \rho$ ⇨ $Tr\,\rho^2 < 1$

(3) Schrödinger evolution does not change the p_k

⇨ $Tr\,\rho^2$ is conserved
pure states stay pure
mixed states stay mixed

Changing pure ⇨ mixed requires non-Hamiltonian evolution – see Cohen Tannoudji D_{III} & E_{III}

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Summary So Far

Density Operator: $\boxed{\rho(t) = \sum_k p_k |\psi_k\rangle\langle\psi_k|}$

Terminology:

$|\psi\rangle$ known → pure state

$p_k, |\psi_k\rangle$ known → mixed state

Properties

(1) $\mathrm{Tr}\,\rho = 1$

(2) $\langle A\rangle = \mathrm{Tr}[\rho A]$

(3) $P(a_n) = \mathrm{Tr}[\rho P_n]$, P_n: projector onto $\mathcal{E}$

(4) $\frac{d}{dt}\rho = \frac{1}{i\hbar}[H, \rho]$ Schrödinger Eq.

(5) ρ pure → $\rho^2 = \rho$, $\mathrm{Tr}\,\rho^2 = 1$

(6) $\frac{d}{dt}\mathrm{Tr}\,\rho^2 = 0$ → S. E. conserves purity

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Separate Description of Part of a System

Let $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$

T.P. Basis $\{|u_i(1)\rangle\} \otimes \{|v_p(2)\rangle\}$

Density Operator ρ in $\mathcal{E}$ ← Describes global system

Goal: To "reverse engineer operators $\rho(1)$ in $\mathcal{E}_1$ and $\rho(2)$ in $\mathcal{E}_2$ such that describe the systems independently

Our starting point is the global density operator

$$\boxed{\rho = \underbrace{\sum_{(ij)(k\ell)}}_{} \rho_{(ij)(k\ell)} \underbrace{|u_i v_j\rangle}_{}\underbrace{\langle u_k v_\ell|}_{}}$$

$i, k \in$ System (1)
$j, l \in$ System (2)

T.P. basis states

End 09-27-2023

Measurement on One Part of a System

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T.P. basis states

Definition: **Partial Trace**

Orthonormal basis in $\mathcal{E}_2$

$$\rho(1) = Tr_2\, \rho = \sum_q \langle v_q | \rho | v_q \rangle$$

$$= \sum_q \sum_{(ij)(kl)} \rho_{(ij)(kl)} \overbrace{\langle v_q || u_i v_j \rangle\langle u_k v_l || v_q \rangle}^{\delta_{jq}\delta_{lq} |u_i\rangle\langle u_k|}$$

$i, k \in$ System (1), $j, l \in$ System (2)

$$= \sum_{ik} \sum_q \rho_{(iq)(kq)} |u_i\rangle\langle u_k| \leftarrow \text{operator in } \mathcal{E}_1$$

Check properties of $\rho(1)$

H.C. c.c. numbers, swap kets & bras

(1) $$\rho(1)^+ = \sum_{ik} \sum_q \rho^*_{(iq)(kq)} |u_k\rangle\langle u_i|$$

$$= \sum_{ik} \sum_q \rho_{(kq)(iq)} |u_k\rangle\langle u_i| \leftarrow \text{Relabel} \quad i \rightarrow k, \; k \rightarrow i$$

$$= \sum_{ik} \sum_q \rho_{(iq)(kq)} |u_i\rangle\langle u_k| = \rho(1)$$

(2) $\rho(1)$ Hermitian → we can choose a basis $\{|w_k(1)\rangle\}$
So $\rho(1)$ is diagonal → $\rho_{(iq)(kq)} \propto \delta_{ik}$

Measurement on One Part of a System

Definition: <u>Partial Trace</u>

$$\rho(1) = \mathrm{Tr}_2\,\rho = \sum_q \langle v_q|\rho|v_q\rangle$$ ← Orthonormal basis in $\mathcal{E}_2$

$$= \sum_q \sum_{(ij)(kl)} \rho_{(ij)(kl)} \underbrace{\langle v_q||u_i v_j\rangle\langle u_k v_l||v_q\rangle}_{\delta_{jq}\delta_{lq}|u_i\rangle\langle u_k|}$$

$i, k \in$ System (1), $j, l \in$ System (2)

$$= \sum_{ik}\sum_q \rho_{(iq)(kq)} |u_i\rangle\langle u_k|$$ ← operator in $\mathcal{E}_1$

Check properties of $\rho(1)$

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(1) $$\rho(1)^\dagger = \sum_{ik}\sum_q \rho^*_{(iq)(kq)} |u_i\rangle\langle u_k|$$

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Thus

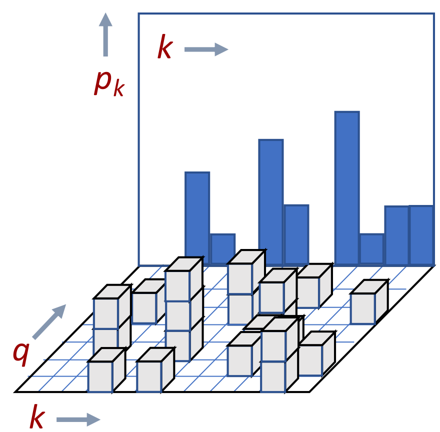
$$\boxed{\rho(1) = \sum_k \underbrace{\sum_q \rho_{(kq)(kq)}}\, |w_k\rangle\langle w_k| = \sum_k p_k |w_k\rangle\langle w_k|}$$

Note:

(1) $\rho_{(kq)(kq)}$ = population of $|w_k(1)\rangle \otimes |v_q(2)\rangle$, i.e. prob. of finding the global system in this state.

(2) $p_k = \sum_q \rho_{(kq)(kq)}$ is a <u>marginal probability</u>, i.e., the prob. of finding system 1 in $|w_k\rangle$, found by adding the probs $\rho_{(kq)(kq)}$ of finding the global system in the states $|w_k v_q\rangle$

Visualization - Marginal Probability



Measurement on One Part of a System

Thus

$$\boxed{\begin{aligned}\rho(1) &= \sum_k \underbrace{\sum_q \rho_{(kq)(kq)}}_{p_k} |w_k\rangle\langle w_k| \\ &= \sum_k p_k |w_k\rangle\langle w_k|\end{aligned}}$$

Note:

$\rho_{(kq)(kq)}$ $\qquad$ $|w_k(1)\rangle \otimes |v_q(2)\rangle$

prob. of finding the global system in this state.

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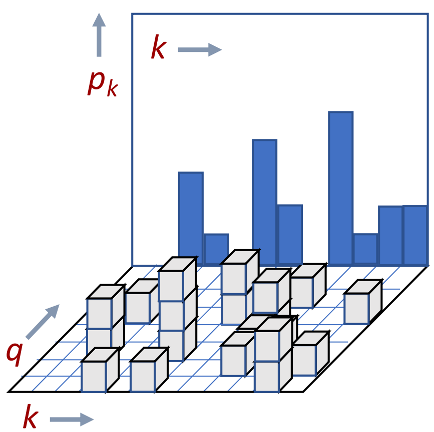

We <u>define</u>

$$\boxed{\begin{aligned}\rho(1) &= Tr_2\, \rho \\ \rho(2) &= Tr_1\, \rho\end{aligned}}$$

Partial Traces
or
Reduced Density Operators

Note: We already know these are Hermitian operators. Also,

$$Tr\, \rho = \sum_n \sum_q \langle u_n v_q|\rho|u_n v_q\rangle$$

global ρ

$$= Tr_1(Tr_2\, \rho) = Tr_1(\rho(1))$$
$$= Tr_2(Tr_1\, \rho) = Tr_2(\rho(2)) \equiv 1$$

Unit Trace Operators!

Expectation Values:

Insert identity here

$$\langle \tilde{A}(1)\rangle = Tr[\rho(1)\tilde{A}(1)] = \sum_{nq} \langle u_n v_q|\rho(1)\tilde{A}(1)|u_n v_q\rangle$$

$$= \sum_{nq}\sum_{n'q'} \underbrace{\langle u_n v_q|\rho(1)|u_{n'} v_{q'}\rangle}_{\rho_{nn'}(1)} \underbrace{\langle u_{n'} v_{q'}|A(1)\otimes \mathbb{1}(2)|u_n v_q\rangle}_{\delta_{qq'}\langle u_{n'}|A(1)|u_n\rangle}$$

identity

$$= \sum_{nn'} \langle u_n|\rho(1)|u_{n'}\rangle\langle u_{n'}|A(1)|u_n\rangle$$

$$= \sum_n \langle u_n|\rho(1)|u_n\rangle = \underline{Tr(\rho(1)A(1))}$$

Measurement on One Part of a System

We **define**

$$\boxed{\begin{aligned} \rho(1) &= Tr_2\,\rho \\ \rho(2) &= Tr_1\,\rho \end{aligned}}$$

Partial Traces
or
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Note: We already know these are Hermitian operators. Also,

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global ρ

$$= Tr_1 (Tr_2\,\rho) = Tr_1 (\rho(1))$$

$$= Tr_2 (Tr_1\,\rho) = Tr_2 (\rho(2)) \stackrel{?}{=} 1$$

Unit Trace Operators!

Expectation Values:

Insert identity here

$$\langle \tilde{A}(1) \rangle = Tr[\rho(1)\tilde{A}(1)] = \sum_{nq} \langle u_n v_q | \rho(1) \tilde{A}(1) | u_n v_q \rangle$$

$$= \sum_{nq} \sum_{n'q'} \underbrace{\langle u_n v_q | \rho(1) | u_{n'} v_{q'} \rangle}_{\rho_{nn'}(1)} \underbrace{\langle u_{n'} v_{q'} | A(1) \otimes \mathbb{1}(2) | u_n v_q \rangle}_{\delta_{qq'} \langle u_{n'} | A(1) | u_n \rangle}$$

identity

$$= \sum_{nn'} \langle u_n | \rho(1) | u_{n'} \rangle \langle u_{n'} | A(1) | u_n \rangle$$

$$= \sum_n \langle u_n | \rho(1) | u_n \rangle = \underline{\underline{Tr(\rho(1) A(1))}}$$

We conclude:

$\rho(1)$, $\rho(2)$ are unit trace, Hermitian Operators

$\langle \tilde{A}(1) \rangle = Tr(\rho(1) A(1))$, $\langle \tilde{B}(2) \rangle = Tr(\rho(2) B(2))$

$\rho(1), \rho(2)$ ⇨ are density operators for system (1) and system (2)

Measurement on One Part of a System

Additional Comments:

(1) If the Global state ≠ T. P. state

➡ Cannot assign states $|\varphi(1)\rangle, |\chi(2)\rangle$ to S_1, S_2

Can assign $\rho(1), \rho(2)$ ➡ Local description

(2) If ρ is pure, $Tr\,\rho = 1$, we still can have

$$Tr\,\rho(1)^2 \neq 1, \quad Tr\,\rho(2)^2 \neq 1$$

(2) If the Global state is a T. P., $|\Phi\rangle = |\varphi(1)\rangle|\chi(2)\rangle$

then
$$\begin{cases} \sigma(1) = |\varphi(1)\rangle\langle\varphi(1)| \\ \tau(2) = |\chi(2)\rangle\langle\chi(2)| \\ \rho = \sigma(1) \otimes \tau(2) \end{cases}$$

(3) The Global state can itself be mixed. In that case a product state will have the following structure

$$\rho = \sigma(1) \otimes \tau(2) \Rightarrow \begin{cases} Tr_2\left[\sigma(1) \otimes \tau(2)\right] = \sigma(1) \\ Tr_1\left[\sigma(1) \otimes \tau(2)\right] = \tau(2) \end{cases}$$

Additional Comments:

(4) However, if $\rho(1) = Tr_2(\rho)$, $\rho(2) = Tr_1(\rho)$

then in general $\rho' = \rho(1) \otimes \rho(2) \neq \rho$

(5) If the evolution of ρ is Hamiltonian, $\dot{\rho} = \frac{1}{i\hbar}[H, \rho]$,
we cannot in general find a $H(1)$ that allows analogous equations for $\rho(1), \rho(2)$

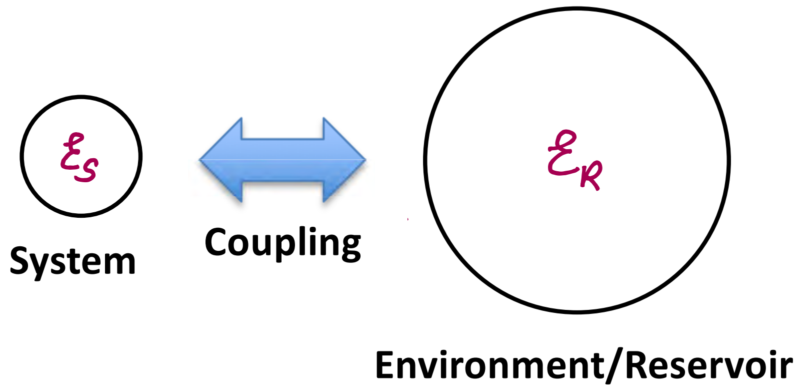
<u>Note</u>:

Hamiltonian evolution conserves the purity of ρ.
However, if $\rho(1)$ is initially pure (unentangled S_1, S_2) the global evolution may entangle S_1, S_2 and cause $\rho(1)$ to become mixed.

Evolution of $\rho(1)$ is not Hamiltonian

Measurement on One Part of a System

Important Application: System-Reservoir Theory



* We do measurements on the system only
  Describe it by ϱ_S, evolve by a non-Hamiltonian Equation of Motion.

* The environment is too large, with too many degrees of freedom to keep track of. Coupling correlates (entangles) the system and environment, but information transferred to the latter is lost.

Important Application: System-Reservoir Theory

* Reasonable assumptions about the environment

  “Master Equation” for ϱ_S

$$\boxed{\dot{\varrho}_S = \frac{1}{i\hbar}[H_S, \varrho_S] + \mathcal{L}(\varrho_S)}$$

* The Liouvillian $\mathcal{L}$
  accounts for relaxation and decoherence

* Alternative description in terms of Decohering Channels.

What comes next ?