Review of Quantum Mechanics

Cohen-Tannoudji Ch. II & III, Preskill 2.1 & 2.3

Review of Quantum Mechanics

Note: Everyone is assumed to be familiar with grad level QM

Quick review focused on 2-level systems, Tensor Product spaces and Density Matrix formalism

State vectors

("Rays" in Preskill)

Unique quantum state $\leftrightarrow$ unique state vector

$|\psi\rangle \in \mathcal{E}$ $\leftarrow$ State Space

Scalar product

$\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^*$

complex number

($\mathcal{E}$ is a Hilbert Space)

Linear Operators

$\forall |\psi\rangle \in \mathcal{E}: \quad A|\psi\rangle = |\psi'\rangle \in \mathcal{E}$

Projectors

$P_\psi = |\psi\rangle\langle\psi|$ $\leftarrow$ Projector on $|\psi\rangle$

$P_{\mathcal{E}_q} = \sum_{i=1}^{q} |\varphi_q^i\rangle\langle\varphi_q^i|$ $\leftarrow$ projector on subspace $\mathcal{E}_q$

Basis in q dimensional $\mathcal{E}_q$

Hermitian Operators

$A^+ = A$

Adjoint $|\psi'\rangle = A|\psi\rangle \leftrightarrow \langle\psi'| = \langle\psi|A^+$

Physical (measurable) quantities !

Review of Quantum Mechanics

Linear Operators

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← Basis in q dimensional $\mathcal{E}_q$

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Physical (measurable) quantities!

Eigenvalue Equation

$A|\psi\rangle = \lambda|\psi\rangle$

A **Hermitian**

- **Eigenvalues of A are real-valued**
- **Eigenvectors** $A|\psi\rangle = \lambda|\psi\rangle$, $A|\varphi\rangle = \mu|\varphi\rangle$ **are orthogonal if** $\lambda \neq \mu$
- **Eigenvectors of A form orthonormal basis in $\mathcal{E}$**

Commuting Observables

$[A,B] \equiv AB - BA = 0$ ⇒

$\exists$ **orthonormal basis in $\mathcal{E}$ of common eigenvectors of A, B**

Review of Quantum Mechanics

Eigenvalue Equation $A|\psi\rangle = \lambda|\psi\rangle$

A Hermitian

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Commuting Observables

$[A,B] \equiv AB - BA = 0$ ⇒

$\exists$ orthonormal basis in $\mathcal{E}$ of common eigenvectors of A, B

C.S.C.O (Complete set of commuting observables)

Set $A, B, C\ldots$ such that basis $\exists$ in $\mathcal{E}$ of eigenvectors $|a_m, b_m, c_m \ldots\rangle$ uniquely labeled by the set of eigenvalues a_m, b_m, c_m

Example H, L^2, L_z for the Hydrogen atom

Unitary Operators

U is unitary ⇒ $U^{-1} = U^{\dagger} \Leftrightarrow U^{\dagger}U = UU^{\dagger} = \mathbb{1}$

Scalar product invariant: $\langle\psi|\varphi\rangle = \langle\psi|U^{\dagger}U|\varphi\rangle$

⇒ U is a change of basis in $\mathcal{E}$

$U|v\rangle = \lambda|v\rangle$ ⇒ $\lambda = e^{i\theta}$

eigenvecs for $\lambda \neq \lambda'$ are orthogonal

Review of Quantum Mechanics

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Representation and bases

The set $\{|u_i\rangle\}$ forms a basis in $\mathcal{E}$ if the expansion

$$|\psi\rangle = \sum_i \langle u_i|\psi\rangle |u_i\rangle$$

is unique and exists $\forall |\psi\rangle \in \mathcal{E}$

States $|\psi\rangle \Leftrightarrow \begin{bmatrix} \vdots \\ \langle u_i|\psi\rangle \\ \vdots \end{bmatrix}$

Operators $A \Leftrightarrow \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$

Review of Quantum Mechanics

Postulates of Quantum Mechanics

(1) At a fixed time t the state of a physical system is defined by specifying a ket $|\psi(t)\rangle$ belonging to the state space $\mathcal{E}$.

(2) Every measurable physical quantity $\mathcal{A}$ is described by an operator A acting in $\mathcal{E}$; this operator is an observable.

(3) The only possible result of a measurement of A physical quantity $\mathcal{A}$ is one of the eigenvalues of the corresponding observable A.

(4) (Discrete non-degenerate spectrum) When the physical quantity $\mathcal{A}$ is measured on A system in the normalized state $|\psi\rangle$, the probability $\mathcal{P}(a_n)$ of obtaining the non-degenerate eigenvalue a_n of the observable A is:

$$\mathcal{P}(a_n) = |\langle a_n|\psi\rangle|^2 = \langle\psi|P_n|\psi\rangle$$

where $|a_n\rangle$ is the normalized eigenvector of A associated with the eigenvalue a_n, and $P = |a_n\rangle\langle a_n|$ is the projector onto $|a_n\rangle$.

Postulates of Quantum Mechanics

(5) If the measurement of the physical quantity $\mathcal{A}$ on the system in state $|\psi\rangle$ gives the result a_n, then the state immediately after the measurement is the normalized projection of $|\psi\rangle$ onto $|a_n\rangle$:

$$|\psi_{after}\rangle = \frac{P_n|\psi\rangle}{\langle\psi|P_n|\psi\rangle}$$

Degenerate case: use projector onto the Subspace associated with a_n.

(6) The time evolution of the state vector $|\psi(t)\rangle$ Is governed by the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

where $H(t)$ is the observable associated with the total energy of the system.

See also Note on the Bayesian Update Rule for "classical" probability distributions

Tensor Products of State Spaces

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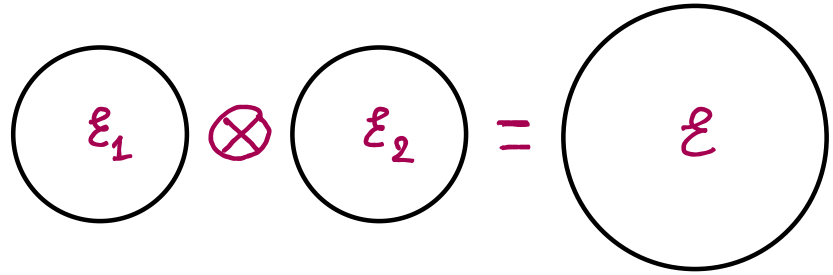
Quantum Mechanics of systems that consist of multiple parts



System 1 System 2 Joint system

Def: Let $\mathcal{E}_1, \mathcal{E}_2$ be vector spaces of dimension N_1, N_2

The vector space $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ is called the

Tensor Product of $\mathcal{E}_1$ and $\mathcal{E}_2$ iff

$\forall$ pairs $|\varphi(1)\rangle \in \mathcal{E}_1, |\chi(2)\rangle \in \mathcal{E}_2$, $\exists$ vector $\in \mathcal{E}$

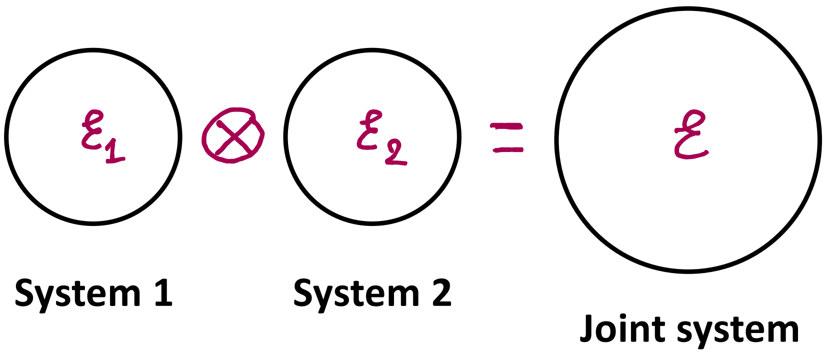
such that

1. The association is linear with respect to multiplication with complex numbers

$$\lambda|\varphi(1)\rangle \otimes \mu|\chi(2)\rangle = \lambda\mu\left[|\varphi(1)\rangle \otimes |\chi(2)\rangle\right]$$

Tensor Products of State Spaces

Quantum Mechanics of systems that consist of multiple parts



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1. The association is linear with respect to multiplication with complex numbers

$$\lambda|\varphi(1)\rangle \otimes \mu|\chi(2)\rangle = \lambda\mu\left[|\varphi(1)\rangle \otimes |\chi(2)\rangle\right]$$

2. Distributive $|\varphi(1)\rangle \otimes \left[a|\chi_1(2)\rangle + b|\chi_2(2)\rangle\right]$

$$= a|\varphi(1)\rangle \otimes |\chi_1(2)\rangle + b|\varphi(1)\rangle \otimes |\chi_2(2)\rangle$$

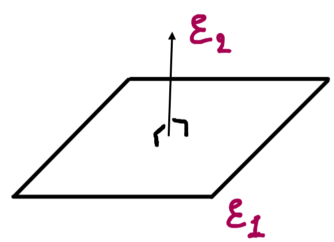
3. Bases $\{|u_i(1)\rangle\}$ in $\mathcal{E}_1$, $\{|v_\ell(2)\rangle\}$ in $\mathcal{E}_2$

$\Rightarrow$ $\{|u_i(1)\rangle \otimes |v_\ell(2)\rangle\}$ is a basis in $\mathcal{E}$

Iff N_1, N_2 are finite, then $Dim(\mathcal{E}) = N_1 \times N_2$

These properties $\Rightarrow$ The usual linear algebra works in $\mathcal{E}$

<u>Analogy</u>: Tensor product of $1D$ & $2D$ geometrical space



<u>Note</u>: $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \neq 3D$ geom. space

SP of vectors in $\mathcal{E}_1$ w/vectors in $\mathcal{E}_2$

<u>not defined</u>

Tensor Products of State Spaces

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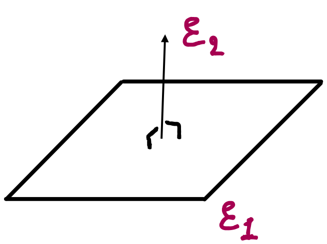

Note: $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \neq 3D$ geom. space

SP of vectors in $\mathcal{E}_1$ w/vectors in $\mathcal{E}_2$ not defined

Vectors in $\mathcal{E}$ Let $|\varphi(1)\rangle = \sum a_i |u_i(1)\rangle$
$|\chi(2)\rangle = \sum b_\ell |v_\ell(2)\rangle$

Then $|\varphi(1)\rangle \otimes |\chi(2)\rangle = \sum_{i,\ell} a_i b_\ell |u_i(1)\rangle \otimes |v_\ell(2)\rangle$

Hugely important:

There are vectors in $\mathcal{E}$ that are not tensor products of vectors from $\mathcal{E}_1, \mathcal{E}_2$

General vector $\in \mathcal{E}$ can be written as

$$|\psi\rangle = \sum_{i,\ell} c_{i\ell} |u_i(1)\rangle \otimes |v_\ell(2)\rangle$$

How to see ? There are $N_1 \times N_2$ prob. ampl's $c_{i\ell}$

These cannot all be written as $a_i \times b_\ell$ where the sets $\{a_i\}, \{b_\ell\}$ are valid probability amplitudes.

Tensor Products of State Spaces

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Example: $\mathcal{E}_1, \mathcal{E}_2$ are qubits, $N_1 = N_2 = 2$

$|\varphi(1)\rangle = a_1 |u_1(1)\rangle + a_2 |u_2(1)\rangle$

$|\chi(2)\rangle = b_1 |v_1(2)\rangle + b_2 |v_2(2)\rangle$

$\Rightarrow$ 2 real-valued variables each

In basis $\{|u_i(1)\rangle \otimes |v_\ell(2)\rangle\}$

Product state $\begin{bmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{bmatrix}$ General state $\begin{bmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{bmatrix}$

4 real-valued variables 6 real-valued variables

N qubits $\Rightarrow$
- product state $\rightarrow$ $2N$ real variables
- general state $\rightarrow$ $2^{N+1} - 2$ real var's

Tensor Products of State Spaces

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N qubits ⇒ product state → $2N$ real variables; general state → $2^{N+1} - 2$ real var's

Note: States $\in \mathcal{E}$ that are not product states are known as

Entangled States or Correlated States

End 09-20-2023

Tensor Products of State Spaces

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N qubits ⇨ $\begin{cases} \text{product state} \rightarrow 2N \text{ real variables} \\ \text{general state} \rightarrow 2^{N+1} - 2 \text{ real var's} \end{cases}$

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Begin 09-25-2023

Back to the Linear Algebra engine of QM

Scalar product: $\left(\langle \varphi'(1)| \otimes \langle \chi'(2)| \right) \left(|\varphi(1)\rangle \otimes |\chi(2)\rangle \right)$

$= \langle \varphi'(1) | \varphi(1) \rangle \langle \chi'(2) | \chi(2) \rangle$

Operators: Let $A(1)$ act in $\mathcal{E}(1)$

The Extension $\tilde{A}(1)$ acting in $\mathcal{E}$ is defined by

$\tilde{A}(1) \left[|\varphi(1)\rangle \otimes |\chi(2)\rangle \right] = \left(A(1) |\varphi(1)\rangle \right) \otimes |\chi(2)\rangle$

Extension $\tilde{B}(2)$ of $B(2)$ into $\mathcal{E}$ is similar

Tensor Products of State Spaces

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Operators: Let $A(1)$ act in $\mathcal{E}(1)$

The **Extension** $\tilde{A}(1)$ acting in $\mathcal{E}$ is defined by

$$\tilde{A}(1)\left[|\varphi(1)\rangle \otimes |\chi(2)\rangle\right] = \left(A(1)|\varphi(1)\rangle\right) \otimes |\chi(2)\rangle$$

Extension $\tilde{B}(2)$ of $B(2)$ into $\mathcal{E}$ is similar

Tensor Product of Operators

$$[A(1) \otimes B(2)][|\varphi(1)\rangle \otimes |\chi(2)\rangle] = [A(1)|\varphi(1)\rangle] \otimes [B(2)|\chi(2)\rangle]$$

$$\Rightarrow A(1) \otimes B(2) = \tilde{A}(1)\tilde{B}(2)$$

special case:
$$\tilde{A}(1) = A(1) \otimes \mathbb{1}(2)$$
$$\tilde{B}(2) = \mathbb{1}(1) \otimes B(2)$$

Commutator

$[\tilde{A}(1), \tilde{B}(2)] = 0$ because $[A(1), \mathbb{1}(1)] = [B(2), \mathbb{1}(2)] = 0$

Notation: Obvious from context

$$|\varphi(1)\rangle \otimes |\chi(2)\rangle \leftrightarrow |\varphi(1)\rangle|\chi(2)\rangle \leftrightarrow |\varphi(1)\chi(2)\rangle$$

$$A(1) \otimes B(2) \leftrightarrow A(1)B(2)$$

$$\tilde{A}(1) \leftrightarrow A(1)$$

Tensor Products of State Spaces

Tensor Product of Operators

$$[A(1)\otimes B(2)][|\varphi(1)\rangle\otimes|\chi(2)\rangle]=[A(1)|\varphi(1)\rangle]\otimes[B(2)|\chi(2)\rangle]$$

$$\Rightarrow\quad A(1)\otimes B(2)=\tilde{A}(1)\tilde{B}(2)$$

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Notation:

Obvious from context

$$|\varphi(1)\rangle\otimes|\chi(2)\rangle\leftrightarrow|\varphi(1)\rangle|\chi(2)\rangle\leftrightarrow|\varphi(1)\chi(2)\rangle$$

$$A(1)\otimes B(2)\leftrightarrow A(1)B(2)$$

$$\tilde{A}(1)\leftrightarrow A(1)$$

Eigenvalue problem in $\mathcal{E}$

Let $A(1)|\varphi_n^i(1)\rangle=a_n|\varphi_n^i(1)\rangle,\ i=1,\dots,g_n\ \Rightarrow$

$$A(1)|\varphi_n^i(1)\chi(2)\rangle=a_n|\varphi_n^i(1)\chi(2)\rangle\quad\forall|\chi(2)\rangle\in\mathcal{E}_2$$

Can choose $|\chi(2)\rangle\in$ orthonormal basis in $\mathcal{E}_2$

$\Rightarrow\ g_i\times N_2$ - fold degeneracy of a_n in $\mathcal{E}$

Furthermore
$$\left.\begin{aligned}A(1)|\varphi_n^i(1)\rangle&=a_n|\varphi_n^i(1)\rangle\\B(2)|\chi_\ell^j(2)\rangle&=b_\ell|\chi_\ell^j(2)\rangle\end{aligned}\right\}\Rightarrow$$

$$(A(1)+B(2))|\varphi_n^i(1)\chi_\ell^j(2)\rangle=(a_n+b_\ell)|\varphi_n^i(1)\chi_\ell^j(2)\rangle$$
$$A(1)B(2)|\varphi_n^i(1)\chi_\ell^j(2)\rangle=a_nb_\ell|\varphi_n^i(1)\chi_\ell^j(2)\rangle$$
$$f(A(1),B(2))|\varphi_n^i(1)\chi_\ell^j(2)\rangle=f(a_n,b_\ell)|\varphi_n^i(1)\chi_\ell^j(2)\rangle$$

Postulates of QM apply in $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}$

$\Rightarrow$ **We are Done!**

Measurement on One Part of a System

Cohen-Tannoudji Ch. III, Complement D_{III}

Measurement on One Part of a System

Quantum Measurement on Bipartite Systems

$$\mathcal{E}_1 \otimes \mathcal{E}_2 = \mathcal{E}$$

System 1 | System 2 | Joint system

Consider the following:

Bipartite System

$$\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$$

$$\tilde{A}(1) = A(1) \otimes \mathbb{1}(2)$$

$A(1)$: Observable on System 1

Possible outcomes when measuring $\tilde{A}(1)$?

$$\left\{ \text{Eigenvalues of } \tilde{A}(1) \right\} = \left\{ \text{Eigenvalues of } A(1) \right\}$$

$\tilde{g}_n = g_n \times N_2$ $\qquad$ g_n

$\Rightarrow$ Same possible outcomes a_n indep of $|\psi\rangle$

Degeneracy in $\mathcal{E}$ increases by a factor N_2

Projector:

$$P_n(1) = \sum_{i=1}^{g_n} |a_n^i(1)\rangle\langle a_n^i(1)|$$

for eigenvalue a_n

Using the recipe to extend an operator into $\mathcal{E}$

$$\tilde{P}_n(1) = P_n(1) \otimes \mathbb{1}(2)$$

$$= \sum_{i=1}^{g_k} \sum_{k} |a_n^i(1) v_k(2)\rangle \langle a_n^i(1) v_k(2)|$$

Probability of outcome a_n, $|\psi\rangle$ general state $\in \mathcal{E}$

$$p(a_n) = \langle\psi|\tilde{P}_n(1)|\psi\rangle$$

$$= \sum_{i=1}^{g_n} \sum_{k} \langle\psi|a_n^i(1) v_k(2)\rangle\langle a_n(1) v_k(2)|\psi\rangle$$

Posterior state $\quad |\psi'\rangle = \dfrac{1}{\sqrt{p(a_n)}} \tilde{P}_n(1)|\psi\rangle$

Measurement on One Part of a System

$\Rightarrow \begin{cases} \text{Same possible outcomes } a_n \text{ indep of } |\psi\rangle \\ \text{Degeneracy in } \mathcal{E} \text{ increases by a factor } N_2 \end{cases}$

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Some Observations:

1. Basis $|v_k(2)\rangle$ arbitrary, no phys. significance

2. Product States. Iff $|\psi\rangle = |\varphi(1)\rangle \otimes |\chi(2)\rangle$ then

$$|\psi'\rangle \propto P_n(1)|\varphi(1)\rangle \otimes \mathbb{1}(2)|\chi(2)\rangle \propto |\varphi'(1)\rangle \otimes |\chi(2)\rangle$$

3. Entangled States. Let all $g_n = 1$

$$|\psi'\rangle = [P_n(1) \otimes \mathbb{1}(2)]|\psi\rangle \propto [P_n(1) \otimes \mathbb{1}(2)] \sum_{n,k} c_{nk} |a_n(1) v_k(2)\rangle$$

$$\propto |a_n(1)\rangle \otimes \sum_k c_{nk} |v_k(2)\rangle = |a_n(1)\rangle \otimes |\chi'(2)\rangle$$

$g_n > 1 \Rightarrow |\psi'\rangle = \sum_{nk} \sum_{i=1}^{g_n} c_{nik} |a_n^i(1) v_k(2)\rangle$

still entangled

Measure C.S.C.O $\Rightarrow |\psi'\rangle = |\varphi'(1)\rangle \otimes |\chi'(2)\rangle$

product state

Measurement on One Part of a System

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Physical Interpretation of T.P. States

From (2) above, measuring $A(1)$, $B(2)$

$$P(a_n, b_k) = \langle\varphi(1)|P_n(1)|\varphi(1)\rangle\langle\chi(2)|P_k(2)|\chi(2)\rangle$$

➡ Outcomes a_n, b_n are Independent Random Var's

Uncorrelated

Physical Interpretation of Entangled States

From (3) above, measuring $A(1)$, $B(2)$

Global $|\psi\rangle$ cannot be written as $|\varphi(1)\rangle \otimes |\chi(2)\rangle$

$$P(a_n, b_k) = \langle\psi|P_n(1)P_k(2)|\psi\rangle$$

In general, a_n & b_k will be correlated random variables

Conclusion: We cannot assign state vectors to the individual subsystems !

Measurement on One Part of a System

Physical Interpretation of T.P. States

From (2) above, measuring $A(1), B(2)$

$$P(a_n, b_k) = \langle \varphi(1) | P_n(1) | \varphi(1) \rangle \langle \chi(2) | P_k(2) | \chi(2) \rangle$$

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In general, a_n & b_k will be correlated random variables

Conclusion: We cannot assign state vectors to the individual subsystems !

Note:

Even though we cannot assign $|\varphi(1)\rangle$, $|\chi(2)\rangle$, it is still possible to have a local description of each subsystem on its own. It must be consistent with tensor product states, yet it must reduce the information that is locally available when the global $|\psi\rangle$ is entangled

Density Matrix Formalism