OPTI 646 **Homework Set #2** **Jessen**

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Alice and Bob play a little game to test their understanding of some basic Quantum Mechanics. Alice prepares a two-level system in a superposition of two orthonormal basis states, $|\psi\rangle = \cos(\theta/2)e^{-i\varphi/2}|+1\rangle + \sin(\theta/2)e^{i\varphi/2}|-1\rangle$, and then gives the system (without revealing the state) to Bob who measures σ_z. Based on the outcome Bob tries to guess what state Alice prepared.

(a) What are the possible outcomes when Bob measures σ_z, and what is the corresponding best guess $|\chi\rangle$ for each outcome.

(b) A measure for the correctness of Bobs guess $|\chi\rangle$ is the fidelity $F = |\langle\psi|\chi\rangle|^2$. What is the fidelity of Bobs guess for each of the possible outcomes of his measurement, as a function of the angles θ and φ chosen by Alice?

(c) In general it is not possible, even in principle, to predict which outcome Bob will get in a given round of the game. For a given $|\psi\rangle$ (i. e. given θ and φ), find the average fidelity that Bob will achieve if he and Alice play the game many times. Sketch its dependence on θ and φ.

(d) Next, assume that Alice draws her two-level system at random from an ensemble with θ and φ uniformly distributed over the intervals $0 \le \theta < \pi$ and $0 \le \varphi < 2\pi$, respectively (Uniform distribution on the Bloch sphere.) She gives the system to Bob who guesses the state as above. What is Bobs expected fidelity averaged over many round of the game?

These integrals will be useful: $\frac{1}{4\pi}\int_0^{\pi}\sin(\theta)d\theta\int_0^{2\pi}d\varphi = 1$, $\frac{1}{4\pi}\int_0^{\pi}\frac{1}{2}\sin^3(\theta)d\theta\int_0^{2\pi}d\varphi = \frac{1}{3}$