# A review of quantum error correction of a qubit encoded in grid states

Jing Wu

December 2020

**Abstract**

Quantum information can be carried by either discrete system for continuous system. In 2001, Gottesman-Kitaev-Preskill (GKP) states are proposed on the analogy of shift-resistant quantum code in discrete case. GKP states are grid states. By encoding a qubit in grid states, we can protect against errors that shift the values of the canonical variables $q$ and $p$. Quantum error correction (QEC) of a qubit encoded in grid states can be realized by employing the conditional displacement gate then doing error correction repeatedly. Researchers in the Yale university showed their recent experiments on QEC of a qubit encoded in grid states.

## 1 Introduction

Quantum information can be carried by either discrete system such as spin or continuous system as Gaussian states. The quantum codes in discrete case are well studied by the general "stabilizer" framework. Starting with a discrete shift-resistant code, Gottesman, Kitaev and Preskill constructed quantum codes for systems described by continuous variable[2]. The code words are superpositions of infinitely squeezed states, a 2D grid of $\delta$ functions in the quadrature plane.

In practice, people work with finite squeezing. The code, $|\mathrm{GKP}\rangle$ state, is described by superpositions of Gaussian functions of width $\Delta$ weighted by a Gaussian envelop function of width $\Delta^{-1}$. This is a square code in the quadrature plane. There are other kinds of grid states such as hexagonal code. Quantum error correction (QEC) is essential to the grid states. Recently, researchers in Yale university proposed QEC protocol and made the experiments for the grid states[1]. In this review, I will discuss the $|\mathrm{GKP}\rangle$ state, its distribution, the QEC protocol for grid states and the results got from people's recent experiments.

## 2 A qubit in grid states

The code words for n-dimension system encoded in an single oscillator can be described by superpositions of infinitely squeezed states up to normalization[2]:

$$\left|\bar{Z} = \omega^j\right\rangle = \sum_{s=-\infty}^{\infty} |q = \alpha(j + ns)\rangle \tag{1}$$

$$\left|\bar{X} = \bar{\omega^j}\right\rangle = \sum_{s=-\infty}^{\infty} \left|p = \frac{2\pi}{n\alpha}(j + ns)\right\rangle \tag{2}$$

where $\alpha$ is chosen to be $\sqrt{\frac{2\pi}{n}}$ to balance $q$ and $p$, and $j = 0, 1, ..., n-1$. It's designed to protect against the shifts with:

$$|\Delta q| < \frac{\alpha}{2} \tag{3}$$

$$|\Delta p| < \frac{\pi}{n\alpha} \tag{4}$$

One can get the Wigner function for the encoded words $\left|\bar{Z} = \omega^j\right\rangle$:

$$W^{(j)}(q,p) = \sum_{s,t} (-1)^{st} \delta\left(p - \frac{\pi}{n\alpha}s\right) \delta\left(q - \alpha j - \frac{n\alpha}{2}t\right) \tag{5}$$

For a qubit ($n = 2$), this is a 2D square grid in the quadrature plane with spacing $\alpha = \sqrt{\pi}$. And we are able to discriminate the error within $\alpha/2 = \sqrt{\pi}/2$ displacement. In practice, people approximate the finitely squeezed states by replacing the $\delta$ function by a normalized Gaussian function of width $\Delta$ and each peak is weighted by another Gaussian function of width $\kappa^{-1}$. To be symmetric between $q$ and $p$, we need to take $\kappa = \Delta$. In the above choices, one can show that the $q$ presentation of code word $j = 0$ is:

$$|\langle q|0\rangle|^2 \approx \frac{2}{\sqrt{\pi}} \sum_{s=-\infty}^{\infty} \exp\{-4\pi\Delta^2 s^2\} \exp\{-(q - 2s\sqrt{\pi})^2/\Delta^2\} \tag{6}$$

which is plotted in figure 1. In general, the code is described by a square grid of Gaussian function in the quadrature plane as plotted in figure 2-(a). The state is called GKP state and can protect the error described by continuous variable. For example, decoherence causes the position $q$ and the momentum $p$ of a particle to diffuse with some nonzero diffusion constant. Then $q$ and $p$ will drift by small amount. People may also use GKP state as a non-Gaussian source for error correction described by continuous variable[3].

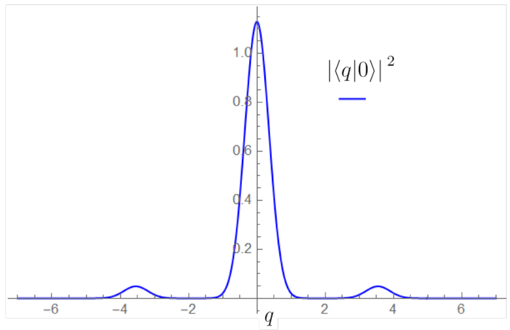


Figure 1: Code word $|0\rangle$ in q presentation for $\Delta = 0.5$.

# 3 QEC protocol of a grid state

People have proposed QEC of a grid state and done the experiments recently[1]. A qubit encoded in grid state is stored in an single oscillator as researchers called the storage oscillator. A superconducting charged qubit is called transmon. In the experiment, conditional displacement is employed to entangle the transmon and the storage oscillator. The phase-space displacements operator of the oscillator is defined as: $\mathrm{D}(\beta) = \exp\{-i\mathrm{Re}(\beta)\boldsymbol{p} + i\mathrm{Im}(\beta)\boldsymbol{q}\}$. Then the conditional displacement describing the coupling is:

$$\mathrm{CD}(\beta) = \exp\left\{i[-\mathrm{Re}(\beta)\boldsymbol{p} + \mathrm{Im}(\beta)\boldsymbol{q}]\frac{\boldsymbol{\sigma_z}}{\boldsymbol{2}}\right\} \tag{7}$$

The coupling can be effectively activated with microwave drives in the presence of the naturally present dispersive interaction[4]. If the transmon is in state $|\uparrow_x\rangle$, this operator leads to the rotation of the transmon by an angle of $\langle \mathrm{D}(\beta)\rangle$. By measuring the state of the transmon, we are able to know whether it's rotated or counter-rotated, thus know whether the grid is moved up or down. Actually, people are able to obtain the expectation value of any displacement operator $\langle \mathrm{D}(\beta)\rangle$ for an arbitrary state of the storage oscillator by conditional displacements embedded within a transmon Ramsey sequence. This lead to the characteristic function $C(\beta) \equiv \langle \mathrm{D}(\beta)\rangle$. In the experiment, $\mathrm{Re}(C(\beta))$ is measured to check the corrected states' distribution in the phase-space.

As shown in the figure 2-(a), at each round, QEC includes correction of $q$ and $p$, two peak-sharpening rounds and two envelope-trimming rounds to prevent spreading of the grid-state peaks and envelope in phase space. Since the measurement of the transmon tells whether the grid is moved up or down, a feedback, a fixed-length displacement opposite that direction is applied. The combination of the back-action of the measurement and the feedback sharpens the peaks of the grid state. Similar measurements of small displacement operators and feedback trim the envelope of the grid states to keep it from drifting and expanding. After this, a measurement of the transmon is done to reset the state of the transom. For example, one may measure it along $x$ axis. If it's $|\uparrow_x\rangle$, then we are done otherwise it's $|\downarrow_x\rangle$, a $\pi/2$ phase shift is applied. These above steps are repeated to push the state of the oscillator to the grid state.

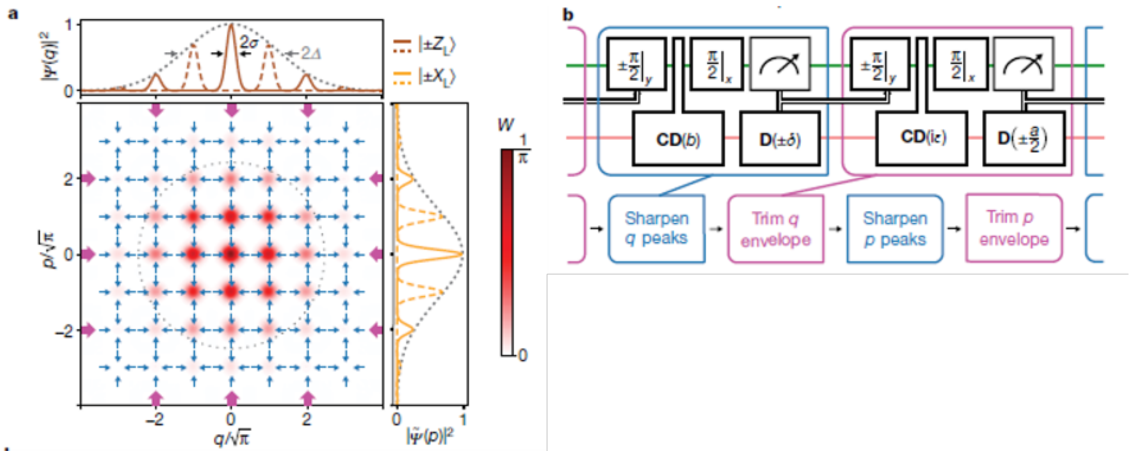


Figure 2: Wigner function and QEC protocol. Figures are from [1]. (a) Simulated Wigner function in phase-space for $\Delta = 0.5$. (b) QEC protocol. Each round involves 4 corrections.

# 4 Experiments

In the experiments, the storage mode has a single-photon lifetime of $T_s = 245\mu s$ and the transmon has energy and coherence lifetimes of $T_1 = 50\mu s$ and $T_{2E} = 60\mu s$. Researchers started from the ground state of the oscillator and applied the protocol indefinitely. Finally, a steady state is reached. And the steady state is checked by its characteristic function. As shown in the figure 3-(a), $\boldsymbol{S}_a = \boldsymbol{D}(a = 2\pi)$ and $\boldsymbol{S}_b = \boldsymbol{D}(b = 2i\pi)$ is measured at each round of correction. They converge rapidly at first 20 round. After some fluctuations, they do not evolve over hundreds of round which haven't shown in the figure. The real part of the characteristic function of the steady state is measured afterwards shown in figure 3-(b). Indeed, the state evolves to the grid state successfully.

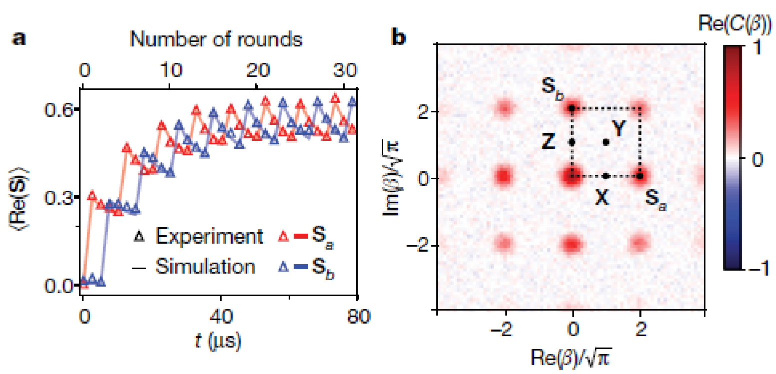


Figure 3: Square code of QEC protocol. Figures are from [1]. (a) Measured average value of $\boldsymbol{S}_a$ and $\boldsymbol{S}_b$ after each round starting from ground state. Each round includes 4 corrections.(b) Measured real part of the characteristic function of the steady state

## References


[1] P. Campagne-Ibarcq, A. Eickbusch, S. Touzard, and et al. Quantum error correction of a qubit encoded in grid states of an oscillator. *Nature*, 584:368–372, 2020.

[2] Daniel Gottesman, Alexei Kitaev, and John Preskill. Encoding a qubit in an oscillator. *Phys. Rev. A*, 64:012310, Jun 2001.

[3] Kyungjoo Noh, S. M. Girvin, and Liang Jiang. Encoding an oscillator into many oscillators. *Phys. Rev. Lett.*, 125:080503, Aug 2020.

[4] A. Wallraff, D. Schuster, A. Blais, and et al. Strong coupling of a single photon to a superconducting qubit using circuit quantum electrodynamics. *Nature*, 431:162–167, 2004.