OPTI 570, Fall 2019 **Midterm Exam 3** Anderson

120-minute written exam to be taken in class:
Wednesday, Nov. 20, 4:00-6:00pm.

Instructions

- You may consult the following items during the exam: a bound copy of the course notes, and anything that you have personally written (on paper or an electronic device) or that you have personally typed and printed. You may use any notes from your studying, and any problems that you have worked. **You may use a calculator, but you may not use its graphing or symbolic manipulation capabilities.** You may **not** use: the course textbook, photocopies of any material (other than the course notes and the homework questions), computers (unless used only to access your notes), or the distributed solutions to the problem sets or practice exam. You may **not** consult other people, or use internet, email, or any other electronic resources. You are also on your honor to refrain from providing any help, hints, or feedback of any sort to other students who are in the process of taking the exam. Violation of these rules will result in a failing grade.
- There are 4 problems on the 3 pages that follow. 100 points are available. Raw scores will be scaled and converted to a final grade.
- Use your own paper to solve all problems. Show enough work that I can follow your reasoning and give you partial credit for problems that are not fully correct.
- It is up to you to convince me that you know how to solve the problems, and to write legibly enough that I do not need to struggle to interpret your work. However, I expect you to work quickly, and that the neatness of your solutions might consequently suffer. That's OK as long as I can interpret your solutions. Draw a box around final answers if your final results are not obvious. If you have a mess of equations all over the page, direct my attention to your line of thought if it is not otherwise obvious. If you have obtained an answer that you know is not correct and you do not have enough time to fix the error, please tell me that you know the answer is wrong, why you know that it is wrong, and guess an appropriate answer – this may help you earn significant partial credit.
- If you are convinced that there is a mistake in a problem, please ask me about it. Alternatively, if I have made an error that is obvious to you, clearly indicate what you think is wrong, what should be changed to make the problem solvable in the manner that you think I intended, then solve the problem. Make sure that I can understand how you have modified the problem to make it solvable. Part of the challenge of learning a new subject is to try to identify mistakes and speculate about the original intention!
- The 120 minutes that you have available for the exam begins as soon as you open the exam.

You are required to turn in the exam questions when you turn in your answers. Before starting the following problems, **make sure your name is written on this page AND on your first sheet of paper.**

1. Hellmann-Feynman Theorem [10 pts.] For an eigenstate $|\psi_k\rangle$ of a time-independent Hamiltonian $\hat{H}$, such that $\hat{H}|\psi_k\rangle = E_k|\psi_k\rangle$, the *Hellmann-Feynman Theorem* states

$$\frac{dE_k}{d\xi} = \langle\psi_k|\,\frac{d\hat{H}}{d\xi}\,|\psi_k\rangle,$$

where the derivatives are taken with respect to some fixed parameter (here represented by ξ) of the specific problem. This theorem is straightforward to prove, but you are not asked to do so here. The theorem can come in handy when calculating certain expectation values. For example, if $\hat{H}_{QHO} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega^2\hat{X}^2$ and the eigenvalues of $\hat{H}_{QHO}$ are $E_n = \hbar\omega(n+\frac{1}{2})$, taking derivatives with respect to ω gives

$$\frac{dE_n}{d\omega} = \hbar(n+1/2) \quad \text{and} \quad \left\langle \frac{d\hat{H}}{d\omega} \right\rangle = m\omega\langle\hat{X}^2\rangle$$

and therefore $\langle\hat{X}^2\rangle = \frac{\hbar}{m\omega}(n+\frac{1}{2})$ for any harmonic oscillator energy eigenstate.

Now consider the Hamiltonian for the spinless hydrogen atom model,

$$\hat{H} = \frac{\hat{P}^2}{2m} - \frac{e^2}{\hat{R}}$$

where m is the electron mass (which we will assume is identical to the reduced mass), $\hat{P}^2$ is the square of the momentum vector operator, $\hat{R}$ is the *magnitude* of the position vector operator, and $e^2 \equiv \frac{q^2}{4\pi\epsilon_0}$ where $q = 1.6\times10^{-19}$ C. Use the Hellmann-Feynman Theorem to evaluate $\langle 1/\hat{R}\rangle$ for any hydrogen energy eigenstate $|n,\, l,\, m_z\rangle$ with energy eigenvalue $E_n = -\frac{me^4}{2\hbar^2 n^2}$. Proceed by first taking derivatives of $\hat{H}$ and E_n with respect to e. Put your final answer in terms of the Bohr radius $a_0 = \frac{\hbar^2}{me^2}$.

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2. [20 pts.] A ^{87}Rb atom (3/2 nuclear spin) is in the $5\,^2D_{3/2}\,|F=2,\, m_F=2\rangle$ state, where $\hbar m_F$ is the $\hat{\mathbf{z}}$-component of the atom's total angular momentum.

(a) If the $\hat{\mathbf{z}}$-component of the nuclear spin is to be measured, what are the possible measurement results, and the associated probabilities of obtaining each result?

(b) If the $\hat{\mathbf{z}}$-component of the total orbital angular momentum of the electrons is to be measured, what is the probability that the result will be zero? Hint: feel free to skip the calculations of quantities that don't help you answer the question asked.

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3. [30 pts.] Consider a particle with a spin quantum number $s = 1/2$ and a negative gyromagnetic ratio γ. For times $t \geq 0$, the particle is in the presence of a spatially uniform magnetic field that points in the $\hat{\mathbf{x}}$ direction. The magnetic field has an exponentially decaying amplitude given by $B_0 \exp\{-\Gamma t\}$. Both B_0 and Γ are real and positive. At time $t = 0$, the particle is in a spin state $|\psi(0)\rangle = |+\rangle_z$ corresponding to spin up along the $\hat{\mathbf{z}}$ direction.

(a) For $t \geq 0$, calculate $P_-(t)$, the time-dependent probability of finding the particle in the state $|-\rangle_z$, which corresponds to spin down along the $\hat{\mathbf{z}}$ direction.

(b) Carefully (and methodically) sketch $P_-(t)$ for the case $\Gamma = -\gamma B_0/(2\pi)$. Remember $\gamma < 0$.

(c) For $\Gamma = -\gamma B_0/(2\pi)$, at what time is P_- maximized, and what is this maximum value?

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4. Neutrino oscillations. [40 pts.] A neutron (n) can interact with an *electron neutrino* (ν_r), producing a proton (p) and an electron (e). This process is characterized by the expression

$$\nu_e \; + \; n \to p \; + \; e.$$

Similarly, if a *muon neutrino* ν_μ interacts with a neutron, a proton and a *muon* (μ) can be produced:

$$\nu_\mu \; + \; n \to p \; + \; \mu.$$

Since $\nu_e \; + \; n \to p \; + \; \mu$ and $\nu_\mu \; + \; n \to p \; + \; e$ do not occur, neutrino types ("flavors") can be determined by examining the particles produced in interactions with neutrons. (A third type of neutrino exists, but we neglect them in this problem.) Neutrinos are also *produced* as one of these types (ν_e or ν_μ) in different processes. Although neutrinos have non-zero mass, the ν_e and ν_μ particles apparently *do not have well-defined mass*! This topic was the subject of the 2015 Physics Nobel Prize. The problem below gives an example of how neutrino mass differences can be measured using a quantum oscillation effect, and challenges our intuition on the meaning of mass. The underlying theory is based on the idea that the ν_e and ν_μ particles actually represent *two different quantum states* of the physical entity that we call a neutrino.

In an accelerator, neutrinos are produced with well-defined momentum p (from here on, p refers to momentum, rather than symbolizing a proton). Since a neutrino's mass m is so small, and since neutrinos travel very nearly at the speed of light c, neutrino energy $E = \sqrt{p^2c^2 + m^2c^4}$ is approximated by $E \simeq pc + \frac{m^2c^4}{2pc}$.

Let $\hat{H}$ be the Hamiltonian of a free neutrino of momentum p. We label the eigenstates of $\hat{H}$ as $|\nu_1\rangle$ and $|\nu_2\rangle$. Suppose that

$$\hat{H}|\nu_1\rangle = E_1|\nu_1\rangle, \quad \hat{H}|\nu_2\rangle = E_2|\nu_2\rangle$$

where

$$E_1 = pc + \frac{m_1^2c^4}{2pc}, \quad E_2 = pc + \frac{m_2^2c^4}{2pc}.$$

Here, m_1 and m_2 are the masses of the two states $|\nu_1\rangle$ and $|\nu_2\rangle$, and we assume $m_1 > m_2$. We can say that $|\nu_1\rangle$ and $|\nu_2\rangle$ are not only energy eigenstates, but also *mass eigenstates* (to make sense of this, we might assume that a mass operator could be constructed). However, these statements do not mean that a particle must necessarily exist at any arbitrary point in time in one of these Hamiltonian's eigenstates, i.e., in a state with definite mass.

Let $|\nu_e\rangle$ and $|\nu_\mu\rangle$ indicate the electron and muon neutrino quantum states, respectively. Based on the processes defined at the beginning of this problem, these are the states that are produced or detected in an experiment, rather than a state of well-defined mass ($|\nu_1\rangle$ or $|\nu_2\rangle$). In other words, we are supposing that neutrino mass is not directly measurable using the interactions described earlier. So we will also suppose the following:

$$|\nu_e\rangle = |\nu_1\rangle \cos\theta + |\nu_2\rangle \sin\theta,$$

$$|\nu_\mu\rangle = -|\nu_1\rangle \sin\theta + |\nu_2\rangle \cos\theta,$$

where θ is some constant real scalar called the mixing angle (which has nothing to do with mixed states). θ just defines a transformation from one basis to another, and is not an angle in space.

(a) Suppose that a muon neutrino of momentum p is produced in a particle accelerator at time $t = 0$, so that we write the neutrino's initial quantum state as $|\nu(t = 0)\rangle = |\nu_\mu\rangle$. Calculate the time-dependent neutrino state $|\nu(t)\rangle$ for times $t > 0$, writing your answer in terms of $|\nu_1\rangle$ and $|\nu_2\rangle$.

(b) What is the probability of detecting the neutrino of part (a) in the state $|\nu_e\rangle$ at a later time t? Express the result in terms of θ, c, p, t, $\hbar$ and $\delta(m^2) \equiv m_1^2 - m_2^2$.

(c) Suppose that after the neutrino described above is produced, it is detected in a target located a distance d from the production point. Express the probability of part (b) in terms of d. Use the approximation that the neutrino travels at the speed of light, c.

(d) You should see that for any θ there will be many distances d where the probability of detecting the neutrino in state $|\nu_e\rangle$ is maximized, given that it was initially produced in state $|\nu_\mu\rangle$. Calculate a value for the shortest distance d where this probability is maximized. Use the following in your calculation: $\delta(m^2) \cdot c^4 = 1\ (\text{eV})^2$, $pc = 10\ \text{GeV} = 10^{10}\ \text{eV}$, and $\hbar = 6.6 \times 10^{-16}\ \text{eV}\cdot\text{s}$.

(e) In 1998, convincing evidence for such neutrino oscillations was reported using atmospheric neutrino data resulting from a 535-day exposure of the Super-Kamiokande detector (Fukuda et al., PRL 81, 1562 (1998)). Other recent experiments with high-precision detectors have also provided evidence for neutrino oscillations. Based on the experimental results, we can realistically suppose that that $\delta(m^2) \cdot c^4 \approx 10^{-3}\ (\text{eV})^2$ (instead of the estimate given above). Using this number, and letting $pc = 10^{10}$ eV and $\theta = \pi/4$, what is the probability of detecting a neutrino in the $|\nu_e\rangle$ state given that it was produced in the $|\nu_\mu\rangle$ state 1000 m away? Your answer should be small but non-zero, and give some insight into why it is difficult to detect neutrino oscillations in lab experiments.

(f) Neutrinos can be produced when cosmic ray protons enter the earth's atmosphere, and it turns out there are twice as many ν_μ particles as ν_e detected. So suppose that a neutrino traveling through the atmosphere reaches the earth's surface in a mixed state with probabilities of 1/3 to be found in the $|\nu_e\rangle$ state, and 2/3 to be found in the $|\nu_\mu\rangle$ state. Write the density matrix for this neutrino in both the $\{|\nu_e\rangle, |\nu_\mu\rangle\}$ representation, and in the $\{|\nu_1\rangle, |\nu_2\rangle\}$ representation in terms of θ. Make sure you label your answers appropriately.

End of exam. Turn in the exam questions along with your answers.