SOLUTIONS

OPTI 570, Fall 2020 **Problem Set 4** Prof. B.P. Anderson

Due: Thurs, Sep. 17.

Problem I.

(a) Using the *sixth postulate* in CT chapter III, show that $\frac{d}{dt}\langle\psi(t)|\phi(t)\rangle = 0$, where $|\psi(t)\rangle$ and $|\phi(t)\rangle$ are arbitrary states in the same state space. Based on this result show that Schrödinger evolution over some time interval $[t_0, t]$ is a *unitary transformation.* (See CT III, eq. D-2, and complement F_{III} if you get stuck).

(b) Let $|\psi(t_0)\rangle = |\phi_n^i\rangle$, where $|\phi_n^i\rangle$ is an eigenvector of the Hamiltonian: $H|\phi_n^i\rangle = E_n|\phi_n^i\rangle$. Assuming that H does not explicitly depend on time, show that

$$|\psi(t)\rangle = e^{-i(E_n/\hbar)(t-t_0)}|\psi(t_0)\rangle$$

is a solution to the Schrödinger Equation (i.e., when you insert $|\psi(t)\rangle$ into the Schrödinger Equation then work through the math and simplify, do you get a result that makes sense?). Since the global phase factor $e^{-i(E_n/\hbar)t}$ has no physical significance, the physical properties of a system described by a quantum state $|\psi(t)\rangle$ of the above form do not change over time, and the eigenstates of H are therefore called *stationary states.*

(c) More generally, show that if $|\psi(t_0)\rangle = \sum_{n,i} c_n^i|\phi_n^i\rangle$, then $|\psi(t)\rangle = \sum_{n,i} c_n^i e^{-i(E_n/\hbar)(t-t_0)}|\phi_n^i\rangle$ is a solution to the Schrödinger Equation. Show that this state is not a stationary state.

Problem II.

CT III, Complement L-III, problem 1.

IMPORTANT: check that the dimensional units of all answers make sense, and that probabilities and expressions of physical quantities are real.

Problem III.

CT III, Complement L-III, problem 12, **parts (a) and (d) ONLY**. Assume $|\psi(0)\rangle$ is normalized. This should be very quick to complete.

Problem IV.

CT III, Complement L-III, problem 14. This is not a particularly quick problem to solve, but it covers many aspects of quantum mechanical formalism and it is worth spending extra time on this problem if needed. If you can solve and understand all of the parts of this problem, you're probably in really good shape by this point in the semester.

Problem V.

This problem considers mathematical relationships relevant to upcoming topics. Although the notation looks intimidating, the actual math involved is not terribly difficult, but you may need to play around with various approaches in order to complete the steps.

We consider a Hamiltonian $H(t)$ that explicitly depends on time through at least one of its terms. This can happen if a particle with a non-zero magnetic dipole interacts with a time-varying magnetic field, for example. Although it may seem counter-intuitive, it is not necessarily the case that the Hamiltonian commutes with itself at different times: $[H(t),\, H(t')]$ is not necessarily equal to zero if $t \neq t'$.

(a) If $[H(t),\, H(t')] \neq 0$ for arbitrary times t and t', show that $[H(t),\, \int_{t_0}^{t} H(t')dt'] \neq 0$.

(b) Define the operator $F(t) = -\frac{i}{\hbar}\int_{t_0}^{t} H(t')dt'$. Derive the following relationships:

$$\left[F(t),\, \frac{d}{dt}F(t)\right] = -(1/\hbar^2)\left[\int_{t_0}^{t} H(t')dt',\, H(t)\right] = (1/\hbar^2)\left[H(t),\, \int_{t_0}^{t} H(t')dt'\right].$$

(c) With $F(t)$ defined as above, show that if $[F(t),\, \frac{d}{dt}F(t)] \neq 0$, then $\frac{d}{dt}(e^{F(t)}) \neq \frac{dF(t)}{dt}\, e^{F(t)}$. Also show that if $[F(t),\, \frac{dF(t)}{dt}] = 0$, then $\frac{d}{dt}(e^{F(t)}) = \frac{dF(t)}{dt}\, e^{F(t)}$.

(d) The Schrödinger evolution operator $U(t, t_0)$ is defined such that $|\psi(t)\rangle = U(t, t_0)\,|\psi(t_0)\rangle$. After substitution of $|\psi(t)\rangle$ into the Schrödinger Equation, this leads to the equation for $U(t, t_0)$:

$$i\hbar\frac{d}{dt}\,U(t, t_0) = H(t)\;U(t, t_0).$$

Show that the evolution operator is given by the expression

$$U(t, t_0) = e^{-\frac{i}{\hbar}\int_{t_0}^{t} H(t')dt'}$$

IF and ONLY IF $[H(t),\, H(t')] = 0$.

There are some OPTIONAL PROBLEMS on the next few pages - you don't need to turn in any of the following problems. They are somewhat challenging and provided in case you have extra time (yeah right!) and would like some more practice.

Problem VI.

CT III, Complement L_{III}, problem 4. Part (c) is challenging, but give it your best shot. Do not use the position or momentum representations in this problem. This problem derives a close connection between the spreading of a free-particle wavepacket and the spreading of a propagating laser beam.

Problem VII: Deriving the Generalized Uncertainty Principle

PART A.

This is a straightforward math and graphing/visualization problem. It is really rather simple, so don't over-analyze it! The point of the problem is to help you build a visual interpretation of mathematical expressions, especially as related to CT Chapter III, complement C_{III}. The relevance of this problem should become clear later in this problem set.

Consider the function $F(\lambda) = a\lambda^2 + c$, where both a and c are real and positive scalars, and λ is a real variable. Consider a second function $G(\lambda) = b\lambda$, where b is a real scalar (positive or negative).

(a) *Qualitatively* plot both $F(\lambda)$ and $G(\lambda)$ vs λ on each of 3 separate sketches so that your plots correspond to and graphically show the following 3 different cases:

(i) $F(\lambda) = G(\lambda)$ at exactly 2 values of λ,

(ii) $F(\lambda) = G(\lambda)$ at exactly one value of λ, and

(iii) $F(\lambda) > G(\lambda)$ for all values of λ.

Your relative magnitudes of the constants do not matter here as long as they meet the conditions listed above. The sign of b is up to you to choose.

(b) For condition (i) above, write a familiar equation for the two values of λ that solve $F(\lambda) = G(\lambda)$. In this equation you should have the expression $b^2 - 4ac$. We will label this quantity D, that is: $D = b^2 - 4ac$. What condition on D (ie, what range of values of D) corresponds to there being two different solutions to $F(\lambda) = G(\lambda)$ where λ is any real number?

(c) Write an inequality for D that would correspond to the plots (ii) and (iii) above, that is: $F(\lambda) \geq G(\lambda)$ for all real λ. This condition will ensure that $a\lambda^2 - b\lambda + c \geq 0$ for all real values of λ. (Note: $D = b^2 - 4ac$ is called the discriminant of the polynomial $a\lambda^2 - b\lambda + c$.)

Part B.

An operator A is termed "anti-Hermitian" if $A = -A^\dagger$.

(a) Show that if an operator F is Hermitian, then $G = iF$ is an anti-hermitian operator.

(b) Show that if an operator G is anti-Hermitian, its eigenvalues are purely imaginary and the expectation value $\langle G \rangle = \langle \psi | G | \psi \rangle$ for any quantum state $|\psi\rangle$ is also purely imaginary.

(c) Show that if A and B are observables, $[A, B]$ is anti-Hermitian, and therefore that $\langle [A, B] \rangle$ is a purely imaginary number.

Part C.

Suppose A and B are observables, and let $|\psi\rangle$ represent a properly normalized quantum state. Define the Hermitian operator C by the expression $[A, B] = iC$. (Although this is not strictly necessary for the following questions, it can be convenient to help with notation.) Also define a new — and not necessarily normalized — quantum state $|\phi\rangle$ as

$$|\phi\rangle = (A + i\lambda B)|\psi\rangle$$

where λ is any real number.

(a) Calculate the scalar product $\langle\phi|\phi\rangle$, showing the result as a second order polynomial in λ. What condition must the discriminant satisfy? (Remember that this polynomial is equal to a scalar product!) Use your answer to show that

$$\left\langle A^2\right\rangle \left\langle B^2\right\rangle \geq \frac{1}{4}\left|\langle[A, B]\rangle\right|^2.$$

(b) Define two new operators,

$$A' = A - \langle A\rangle \quad \text{and} \quad B' = B - \langle B\rangle.$$

Show that $[A', B'] = [A, B]$ and therefore that

$$\left\langle (A')^2\right\rangle \left\langle (B')^2\right\rangle \geq \frac{1}{4}\left|\langle[A, B]\rangle\right|^2.$$

(c) Show that

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\left|\langle[A, B]\rangle\right|^2.$$

where ΔA is the r.m.s deviation or standard deviation of A (and similarly for B). This result is the most general form of Heisenberg's uncertainty principle. As you can see, it was derived for **any** two observables A and B.

(d) Using the results from above, write Heisenberg's uncertainty principle for the specific case of any two observables Q and P that have the commutation relation $[Q, P] = i\hbar$.

$\boxed{I}$ (a) Postulate VI : $i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)\,|\psi(t)\rangle$

$$\Rightarrow -i\hbar \frac{d}{dt}\langle\psi(t)| = \langle\psi(t)|\hat{H}(t)$$

Similarly, $i\hbar \frac{d}{dt}|\varphi(t)\rangle = \hat{H}(t)\,|\varphi(t)\rangle$

So: $\frac{d}{dt}\langle\psi(t)|\varphi(t)\rangle = \left(\frac{d}{dt}\langle\psi(t)|\right)\cdot|\varphi(t)\rangle + \langle\psi(t)|\left(\frac{d}{dt}|\varphi(t)\rangle\right)$

$$= \frac{i}{\hbar}\langle\psi(t)|\hat{H}(t)|\varphi(t)\rangle - \frac{i}{\hbar}\langle\psi(t)|\hat{H}(t)|\varphi(t)\rangle$$

$= 0 \Rightarrow \langle\psi(t)|\varphi(t)\rangle$ constant in time

If Sch. evolution over some time interval $[t, t_0]$ is a unitary transformation, then $\hat{U}(t,t_0)$ is a unitary operator in the expressions

$$|\psi(t)\rangle = \hat{U}(t,t_0)\,|\psi(t_0)\rangle$$
$$|\varphi(t)\rangle = \hat{U}(t,t_0)\,|\varphi(t_0)\rangle$$

NOTE: IN GENERAL, WE CAN NOT SAY $\hat{U}(t,t_0) = e^{-\frac{i\hat{H}(t)(t-t_0)}{\hbar}}$

Since $\langle\psi(t)|\varphi(t)\rangle$ is const. in time, $\langle\psi(t)|\varphi(t)\rangle = \langle\psi(t_0)|\varphi(t_0)\rangle$

$\langle\psi(t)|\varphi(t)\rangle = \langle\psi(t_0)|\hat{U}^\dagger(t,t_0)\,\hat{U}(t,t_0)\,|\varphi(t_0)\rangle$ for all $|\psi\rangle, |\varphi\rangle$

$$= \langle\psi(t_0)|\varphi(t_0)\rangle \Rightarrow \hat{U}^\dagger\hat{U} = \hat{\mathbb{1}}$$

$\Rightarrow \hat{U}$ is unitary

(b) $|\psi(t_0)\rangle = |\varphi_n^i\rangle$, where $\hat{H}\,|\varphi_n^i\rangle = E_n\,|\varphi_n^i\rangle$

Does $|\psi(t)\rangle = e^{-iE_n(t-t_0)/\hbar}\,|\psi(t_0)\rangle$ solve the S.E.?

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = i\hbar\left(-\frac{iE_n}{\hbar}\right)e^{-iE_n(t-t_0)/\hbar}\,|\psi(t_0)\rangle = E_n\,e^{-iE_n(t-t_0)/\hbar}\,|\varphi_n^i\rangle$$

$$= e^{-iE_n(t-t_0)/\hbar}\left(E_n|\varphi_n^i\rangle\right) = e^{-iE_n(t-t_0)/\hbar}\,\hat{H}\,|\varphi_n^i\rangle$$

$$= \hat{H}\,e^{-iE_n(t-t_0)/\hbar}\,|\varphi_n^i\rangle = \hat{H}\,e^{-iE_n(t-t_0)/\hbar}\,|\psi(t_0)\rangle$$

$\Rightarrow \boxed{i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle = \text{S.E.}}$ ✓

(c) $|\psi(t_0)\rangle = \sum_{n,i} c_n^i |\varphi_n^i\rangle$. Is $|\psi(t)\rangle = \sum_{n,i} c_n^i e^{-iE_n(t-t_0)/\hbar} |\varphi_n^i\rangle$ a soln. to the S.E. ?

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \sum_{n,i} i\hbar c_n^i \left(\frac{-iE_n}{\hbar}\right) e^{-iE_n(t-t_0)/\hbar} |\varphi_n^i\rangle$$

$$= \sum_{n,i} c_n^i E_n e^{-iE_n(t-t_0)/\hbar} |\varphi_n^i\rangle$$

$$= \hat{H} \sum_{n,i} c_n^i e^{-iE_n(t-t_0)/\hbar} |\varphi_n^i\rangle$$

$$= \hat{H} |\psi(t)\rangle$$

$\Rightarrow |\psi(t)\rangle$ solves the S.E.

If $|\psi(t)\rangle$ is stationary, then expectation values are constant in time. Otherwise, $|\psi(t)\rangle$ is not stationary.

Consider $\langle \hat{A} \rangle = \langle \psi(t) | \hat{A} | \psi(t) \rangle$ where $\hat{A}$ is an observable.

$$= \sum_{\substack{m,j \\ n,k}} c_m^{j*} e^{iE_m(t-t_0)/\hbar} \langle \varphi_m^j | \hat{A} | \varphi_n^k \rangle c_n^k e^{-iE_n(t-t_0)/\hbar}$$

$$= \sum_{\substack{m,j \\ n,k}} c_m^{j*} c_n^k e^{-i(E_n - E_m)(t-t_0)/\hbar} \langle \varphi_m^j | \hat{A} | \varphi_n^k \rangle$$

time-dependent. Leads to terms that vary sinusoidally in time.

Example: $|\psi(t)\rangle = \frac{1}{\sqrt{2}}(|\varphi_1\rangle + |\varphi_2\rangle) \Rightarrow c_1 = c_2 = 1/\sqrt{2}$

(implied: no other states share the same energy eigenvalue w/ $|\varphi_1\rangle$, or with $|\varphi_2\rangle$)

$$\Rightarrow \langle \hat{A} \rangle = \Big[\tfrac{1}{2} e^{-i(E_1-E_2)(t-t_0)/\hbar} \langle \varphi_2 | \hat{A} | \varphi_1 \rangle + \tfrac{1}{2} e^{i(E_1-E_2)} \langle \varphi_2 | \hat{A} | \varphi_1 \rangle^*$$

$$+ \tfrac{1}{2} \langle \varphi_1 | \hat{A} | \varphi_1 \rangle + \tfrac{1}{2} \langle \varphi_2 | \hat{A} | \varphi_2 \rangle \Big]$$

$$= \tfrac{1}{2} \langle \varphi_1 | \hat{A} | \varphi_1 \rangle + \tfrac{1}{2} \langle \varphi_2 | \hat{A} | \varphi_2 \rangle + \mathrm{Re}\left\{ e^{-i(E_1-E_2)(t-t_0)/\hbar} \langle \varphi_2 | \hat{A} | \varphi_1 \rangle \right\}$$

oscillating term, time-dependent

III

CT III #1

$$\psi(x) = \frac{N e^{ip_0 x/\hbar}}{\sqrt{x^2+a^2}}$$

(a) $1 = \int dx\, \psi^*(x)\psi(x)$

$$= |N|^2 \int_{-\infty}^{\infty} dx \frac{1}{x^2+a^2} = |N|^2 \frac{1}{a}\int \frac{dx}{a}\frac{1}{(x/a)^2+1} = \frac{|N|^2}{a}\int du\left(\frac{1}{u^2+1}\right)$$

$$= \frac{|N|^2}{a}\Big[\arctan(u)\Big]_{-\infty}^{\infty} = \frac{|N|^2}{a}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = \frac{\pi}{a}|N|^2$$

$\Rightarrow \boxed{N = \sqrt{\frac{a}{\pi}}}$ letting N be real, positive

common mistake: π instead of $\pi/2$

(b) $$P\left(-\frac{a}{\sqrt{3}} < x < \frac{a}{3}\right) = \int_{-a/\sqrt{3}}^{a/\sqrt{3}} \psi^*(x)\psi(x)\,dx$$

$$= \frac{a}{\pi}\int_{-a/\sqrt{3}}^{a/\sqrt{3}} dx \frac{1}{x^2+a^2} = \frac{a}{\pi}\frac{1}{a}\int_{-1/\sqrt{3}}^{1/\sqrt{3}} du\left(\frac{1}{u^2+1}\right)$$

$$= \frac{1}{\pi}\Big[\mathrm{atan}(u)\Big]_{-1/\sqrt{3}}^{1/\sqrt{3}} = \frac{1}{\pi}\left[\frac{\pi}{6}+\frac{\pi}{6}\right] = \boxed{\frac{1}{3}}$$

(c) $$\langle\hat{p}\rangle = \langle\psi|\hat{p}|\psi\rangle = \int_{-\infty}^{\infty}\psi(x)^*\left(-i\hbar\frac{d}{dx}\right)\psi(x)\,dx$$

$$= \int_{-\infty}^{\infty}\psi^*(x)(-i\hbar)(ip_0/\hbar)\psi(x)\,dx$$

$$+ \int_{-\infty}^{\infty}\psi^*(x)(-i\hbar)\left(-\frac{1}{2}\frac{2x}{x^2+a^2}\right)\psi(x)\,dx$$

$$= p_0\underbrace{\int_{-\infty}^{\infty}|\psi|^2dx}_{=1} + i\hbar\,\cancel{\int_{-\infty}^{\infty}\underbrace{|\psi|^2\frac{1}{x^2+a^2}}_{\text{even}}\underbrace{x}_{\text{odd}}\,dx}^{\,0}$$

$$\boxed{\langle\hat{p}\rangle = p_0}$$

$\boxed{\text{III}}$ CT III, # 12

ONLY PARTS (a) & (d) are required. But solutions to all parts are included here

$$V(x) = \begin{cases} 0 & 0 \le x \le a \\ \infty & \text{otherwise} \end{cases}$$

$$\hat{H}|\varphi_n\rangle = E_n|\varphi_n\rangle \quad \text{where} \quad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad n \ge 1$$

$$|\psi(0)\rangle = a_1|\varphi_1\rangle + a_2|\varphi_2\rangle + a_3|\varphi_3\rangle + a_4|\varphi_4\rangle$$

ASSUME NORMALIZATION (otherwise, replace all a_i's below with $a_i / \left(|a_1|^2+|a_2|^2+|a_3|^2+|a_4|^2\right)^{1/2}$)

(a) Possible results of measuring energy

POSTULATE III : $E_1 = \frac{\pi^2\hbar^2}{2ma^2}$, $E_2 = \frac{4\pi^2\hbar^2}{2ma^2}$, $E_3 = \frac{9\pi^2\hbar^2}{2ma^2}$, $E_4 = \frac{16\pi^2\hbar^2}{2ma^2}$

$$\Rightarrow P\left(E < 3\pi^2\hbar^2/ma^2 = 6\pi^2\hbar^2/2ma^2\right)$$

$$= \langle\psi(0)|\left[|\varphi_1\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_2|\right]|\psi(0)\rangle$$

$$= |a_1|^2 + |a_2|^2$$

(b) $\langle\hat{H}\rangle = \langle\psi|\hat{H}|\psi\rangle$

$$= \langle\psi|\left(a_1E_1|\varphi_1\rangle + a_2E_2|\varphi_2\rangle + a_3E_3|\varphi_3\rangle + a_4E_4|\varphi_4\rangle\right)$$

$$= |a_1|^2E_1 + |a_2|^2E_2 + |a_3|^2E_3 + |a_4|^2E_4$$

$$= \frac{\hbar^2\pi^2}{2ma^2}\left[|a_1|^2 + 4|a_2|^2 + 9|a_3|^2 + 16|a_4|^2\right]$$

$$= \underbrace{\frac{\hbar^2\pi^2}{2ma^2}}_{E_1}\sum_{j=1}^{4}|a_j|^2(j^2)$$

(various ways to write this answer)

$$\boxed{\langle\hat{H}\rangle = E_1\sum_{j=1}^{4}|a_j|^2 j^2}$$

$$\langle\hat{H}\rangle^2 = E_1^2\sum_{j,k=1}^{4}|a_j|^2|a_k|^2 j^2 k^2$$

To get $\langle \hat{H}^2 \rangle$

Since $\hat{H}|\varphi_n\rangle = E_n|\varphi_n\rangle \Rightarrow \hat{H}^2|\varphi_n\rangle = E_n^2|\varphi_n\rangle$

$$\Rightarrow \langle \hat{H}^2 \rangle = |a_1|^2 E_1^2 + |a_2|^2 E_2^2 + |a_3|^2 E_3^2 + |a_4|^2 E_4^2$$

$$= \left(\frac{\hbar^2\pi^2}{2ma^2}\right)^2 \sum_{j=1}^{4} |a_j|^2 (j^4) = E_1^2 \sum_{j=1}^{4} |a_j|^2 j^4$$

$$\Delta\hat{H} = \left(\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2 \right)^{1/2}$$

$$= E_1 \left(\sum_{j=1}^{4} |a_j|^2 j^4 - \sum_{j,k=1}^{4} |a_j|^2 |a_k|^2 j^2 k^2 \right)^{1/2}$$

$$= E_1 \left(\sum_{j=1}^{4} |a_j|^2 j^2 \left(j^2 - \sum_{k=1}^{4} |a_k|^2 k^2 \right) \right)^{1/2}$$

(c) $|\psi(t)\rangle = a_1 e^{-iE_1t/\hbar}|\varphi_1\rangle + a_2 e^{-iE_2t/\hbar}|\varphi_2\rangle + a_3 e^{-iE_3t/\hbar}|\varphi_3\rangle + a_4 e^{-iE_4t/\hbar}|\varphi_4\rangle$

Since $\frac{d}{dt}\langle \hat{H} \rangle = \frac{1}{i\hbar}\langle \overbrace{[\hat{H},\hat{H}]}^{0} \rangle = 0$

$\Rightarrow \langle \hat{H} \rangle$ is indep. of time.

Similarly, since $[\hat{H}^2, \hat{H}] = 0$,

$\langle \hat{H}^2 \rangle$ is indep. of time

$\rightarrow$ yes, these results stay valid for all times

(d) If $E = 8\pi^2\hbar^2/ma^2 = 16\pi^2\hbar^2/2ma^2 = E_4$

then $|\psi(t > t_{meas})\rangle = e^{-iE_4t/\hbar}|\varphi_4\rangle$

after the measurement,

Further energy measurements on this state will give the result $16\pi^2\hbar^2/2ma^2$

IV

CT III, #14

$$H = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad A = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad B = b\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in $\{|u_i\rangle\}$ representation

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$$

($|\psi\rangle$ is normalized to 1)

(a) Measure energy.

Outcomes	Prob.
$\hbar\omega_0$	$1/2$
$2\hbar\omega_0$	$1/4 + 1/4 = 1/2$

Explicitly: $P(\hbar\omega_0) = \langle\psi(0)|u_1\rangle\langle u_1|\psi(0)\rangle = 1/2$

$P(2\hbar\omega_0) = \langle\psi(0)|\left(|u_2\rangle\langle u_2| + |u_3\rangle\langle u_3|\right)|\psi(0)\rangle = 1/2$

$$\langle H\rangle = \hbar\omega_0 P(\hbar\omega_0) + 2\hbar\omega_0 P(2\hbar\omega_0) = \frac{1}{2}\hbar\omega_0 + \frac{1}{2}(2\hbar\omega_0)$$
$$= (3/2)\hbar\omega_0$$

(or, use $\langle H\rangle = \langle\psi(0)|H|\psi(0)\rangle$
with $H = \hbar\omega_0\left(|u_1\rangle\langle u_1| + 2|u_2\rangle\langle u_2| + 2|u_3\rangle\langle u_3|\right)$)

$$H^2 = [\hbar\omega_0]^2[\,|u_1\rangle\langle u_1| + 2|u_2\rangle\langle u_2| + 2|u_3\rangle\langle u_3|\,]^2$$
$$= (\hbar\omega_0)^2(\,|u_1\rangle\langle u_1| + 4|u_2\rangle\langle u_2| + 4|u_3\rangle\langle u_3|\,)$$
$$\langle H\rangle^2 = \langle\psi(0)|H^2|\psi(0)\rangle = \frac{1}{2}(\hbar\omega_0)^2 + (\hbar\omega_0)^2 + (\hbar\omega_0)^2 = \frac{5}{2}(\hbar\omega_0)^2$$

$$\Rightarrow \Delta H = (\langle H^2\rangle - \langle H\rangle^2)^{1/2}$$
$$= \left[5/2(\hbar\omega_0)^2 - \left(\frac{3}{2}\right)^2(\hbar\omega_0)^2\right]^{1/2} = \left[\frac{1}{4}(\hbar\omega_0)^2\right]^{1/2}$$
$$= \frac{1}{2}\hbar\omega_0$$

(b) Measure A at $t=0$

Possible results: e-vals of A (actually, A/a)

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0 = (1-\lambda)(\lambda^2-1)$$

$\downarrow$ $\lambda = 1$ $\quad$ $\downarrow$ $\lambda = \pm 1$

$\Rightarrow$ $\lambda = 1$ doubly degenerate
$\lambda = -1$ non-degenerate

$\Rightarrow$ e vals $a, a, -a$

e-vectors

$\underline{\lambda = 1}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = +1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \\ y \end{pmatrix}$$

$\left(\frac{A}{a}\right)|e_1\rangle = |e_1\rangle$ $\quad$ $\uparrow$ just evaluating $\left(\frac{A}{a}\right)|e_1\rangle$

one possibility: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \equiv |a_1\rangle$

another orthogonal possibility: $\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \equiv |a_2\rangle$

$\underline{\lambda = -1}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \\ y \end{pmatrix}$$

$\Rightarrow$ $x = x$, $y = -z$

$$|a_3\rangle \equiv \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$\Rightarrow$ $\langle a_i | a_j \rangle = \delta_{ij}$

$\Rightarrow$ $|a_1\rangle = |u_1\rangle$

$$\left.\begin{aligned} |a_2\rangle &= \tfrac{1}{\sqrt{2}}|u_2\rangle + \tfrac{1}{\sqrt{2}}|u_3\rangle \\ |a_3\rangle &= \tfrac{1}{\sqrt{2}}|u_2\rangle - \tfrac{1}{\sqrt{2}}|u_3\rangle \end{aligned}\right\} \quad \begin{aligned} |u_2\rangle &= \tfrac{1}{\sqrt{2}}\left(|a_2\rangle + |a_3\rangle\right) \\ |u_3\rangle &= \tfrac{1}{\sqrt{2}}\left(|a_2\rangle - |a_3\rangle\right) \end{aligned}$$

<u>However</u>, you should note that some of this work was not necessary.

Consider again

$$\frac{H}{\hbar\omega_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \frac{A}{a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

By inspection, you should be able to see that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector of <u>both</u> $\frac{H}{\hbar\omega_0}$ and $\frac{A}{a}$.

Furthermore, if we wanted to see whether these matrices commuted with each other, or to find common eigenvectors if they did, we could examine the subspace spanned by the basis vectors <u>other</u> than $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Call this subspace $\mathcal{E}_2$. It is spanned by $\{|u_2\rangle, |u_3\rangle\}$.

In this subspace $\mathcal{E}_2$

$$\frac{H^{(2)}}{\hbar\omega_0} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \frac{A^{(2)}}{a} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

IF $\left[\frac{H^{(2)}}{\hbar\omega_0}, \frac{A^{(2)}}{a}\right] = 0$, then $[H,A] = 0$ for the full state space.

This is indeed the case, since $H^{(2)}$ is proportional to the identity matrix.

$\Rightarrow [H,A] = 0$, and there are common eigenvectors of $A + H$.

There are also common eigenvectors of $\frac{H^{(2)}}{\hbar\omega_0}$ + $\frac{A^{(2)}}{a}$.

Call these $\begin{pmatrix} e_1 \\ f_1 \end{pmatrix}$, $\begin{pmatrix} e_2 \\ f_2 \end{pmatrix}$ $\rightarrow$ They are orthogonal.

You should be able to see by inspection that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ will be orthogonal to $\begin{pmatrix} 0 \\ e_1 \\ f_1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ e_2 \\ f_2 \end{pmatrix}$.

Also, that $\begin{pmatrix} 0 \\ e_1 \\ f_1 \end{pmatrix}$ is orthogonal to $\begin{pmatrix} 0 \\ e_2 \\ f_2 \end{pmatrix}$.

And finally, that $\begin{pmatrix} 0 \\ e_1 \\ f_1 \end{pmatrix}$ + $\begin{pmatrix} 0 \\ e_2 \\ f_2 \end{pmatrix}$ will be the last two eigenvectors of $H + A$.

(If you can't tell all of this, prove that it must be true.)

This is a long-winded way of saying that since we can immediately see that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector of $H+A$, we reduce our full problem to one over a 2-dimensional subspace.

$\Rightarrow$ Part (b)'s solution could have gone like this:

evals of A/a:

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 = \lambda^2 - 1 \Rightarrow \lambda = \pm 1$$

evectors: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ f_1 \end{pmatrix} = \begin{pmatrix} e_1 \\ f_1 \end{pmatrix} = \begin{pmatrix} f_1 \\ e_1 \end{pmatrix}$

$\Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the normalized solution for e-val $\lambda = +1$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ f_1 \end{pmatrix} = -\begin{pmatrix} e_1 \\ f_1 \end{pmatrix} = \begin{pmatrix} f_1 \\ e_1 \end{pmatrix}$

$\Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is the normalized solution (for $\lambda = -1$)

$\Rightarrow |a_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |u_1\rangle$

$|a_2\rangle = \frac{1}{\sqrt{2}}|u_2\rangle + \frac{1}{\sqrt{2}}|u_3\rangle = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$|a_3\rangle = \frac{1}{\sqrt{2}}|u_1\rangle - \frac{1}{\sqrt{2}}|u_3\rangle = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

Whether following IV-2 or IV-3,4, we conclude:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}(|u_2\rangle + |u_3\rangle) = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|u_2\rangle + \frac{1}{\sqrt{2}}|u_3\rangle\right)$$
$$= \frac{1}{\sqrt{2}}|a_1\rangle + \frac{1}{\sqrt{2}}|a_2\rangle$$

Measure A.	Outcomes	Prob.
	a	$\frac{1}{2} + \frac{1}{2} = 1$
	$-a$	0

Or calculate it all out explicitly from the $\{|u\rangle\}$ basis

$$P(a) = \langle\psi(0)|\left[|u_1\rangle\langle u_1| + \frac{1}{2}(|u_2\rangle + |u_3\rangle)(\langle u_2| + \langle u_3|)\right]|\psi(0)\rangle$$
$$= \langle\psi(0)|\left[|u_1\rangle\langle u_1| + \frac{1}{2}|u_2\rangle\langle u_2| + \frac{1}{2}|u_3\rangle\langle u_3| + \frac{1}{2}|u_2\rangle\langle u_3| + \frac{1}{2}|u_3\rangle\langle u_2|\right]|\psi(0)\rangle$$
$$= \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1$$

$$P(-a) = \langle\psi(0)|a_3\rangle\langle a_3|\psi(0)\rangle$$
$$= \langle\psi(0)|\ \frac{1}{2}(|u_2\rangle - |u_3\rangle)(\langle u_2| - \langle u_3|)\ |\psi(0)\rangle$$
$$= \langle\psi(0)|\ \frac{1}{2}\left(|u_2\rangle\langle u_2| + |u_3\rangle\langle u_3| - |u_2\rangle\langle u_3| - |u_3\rangle\langle u_2|\right)|\psi(0)\rangle$$
$$= \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4}\right) = 0$$

With probability of 1, the state is projected onto the subspace spanned by $\{|a_1\rangle, |a_2\rangle\}$. $\Rightarrow$ $|\psi(0)\rangle$ is already in this subspace, so the measurement has no effect on altering the state of the system.

$\Rightarrow$ $|\psi(0)\rangle$ is still the state of the system.

(c) $|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega_0 t} |u_1\rangle + \frac{1}{2} e^{-i2\omega_0 t}\left(|u_2\rangle + |u_3\rangle\right)$

Vector notation $|\psi(t)\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\omega_0 t} \\ \frac{1}{2} e^{-i2\omega_0 t} \\ \frac{1}{2} e^{-i2\omega_0 t} \end{pmatrix}$ in the $\{|u_i\rangle\}$ basis

(d) $\langle A \rangle = \langle \psi(t) | A | \psi(t) \rangle$

$$= a \underbrace{\left(\frac{1}{\sqrt{2}} e^{i\omega_0 t}, \frac{1}{2} e^{-i2\omega_0 t}, \frac{1}{2} e^{-i2\omega_0 t}\right) \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\omega_0 t} \\ \frac{1}{2} e^{-i2\omega_0 t} \\ \frac{1}{2} e^{-i2\omega_0 t} \end{pmatrix}}_{\begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\omega_0 t} \\ \frac{1}{2} e^{-i2\omega_0 t} \\ \frac{1}{2} e^{-i2\omega_0 t} \end{pmatrix}}}_{=1}$$

$\Rightarrow \langle A \rangle = a$

$$\langle B \rangle = \langle \psi(t) | B | \psi(t) \rangle = b\left(\frac{e^{i\omega_0 t}}{\sqrt{2}}, \frac{e^{+i2\omega_0 t}}{2}, \frac{e^{2i\omega_0 t}}{2}\right) \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\omega_0 t} \\ \frac{1}{2} e^{-i2\omega_0 t} \\ \frac{1}{2} e^{-i2\omega_0 t} \end{pmatrix}}_{\begin{pmatrix} \frac{1}{2} e^{-i2\omega_0 t} \\ \frac{1}{\sqrt{2}} e^{-i\omega_0 t} \\ \frac{1}{2} e^{-i2\omega_0 t} \end{pmatrix}}$$

$$= b\left[\frac{1}{2\sqrt{2}} e^{-i\omega_0 t} + \frac{1}{2\sqrt{2}} e^{+i\omega_0 t}\right] + \frac{b}{4}$$

$$\langle B \rangle = \frac{b}{\sqrt{2}} \cos(\omega_0 t) + \frac{b}{4}$$

<u>comments</u> $\langle A \rangle$ is time independent

$$\Rightarrow \frac{d}{dt}\langle A \rangle = 0$$

$$= \frac{1}{i\hbar}\langle [A, H] \rangle + \langle \partial A/\partial t \rangle^{0}$$

$$\Rightarrow [A, H] = 0$$

check : $HA = a\hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} a\hbar\omega_0$

$AH = a\hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} a\hbar\omega_0 = HA$ ✓

Similarly, $\langle B \rangle$ is time <u>dependent</u>,

$HB = b\hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} b\hbar\omega_0$

$BH = b\hbar\omega_0 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} b\hbar\omega_0$

$$\Rightarrow [B, H] \neq 0$$

Consistent w/ eqs of motion for $\langle A \rangle$, $\langle B \rangle$

(e) Measuring A at time t $\rightarrow$ same outcomes $(a, -a)$. A is a constant of the motion

Measurement of B : outcomes and probabilities will be time dependent.

First solve $\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(\lambda^2 - 1) = 0$

$\Rightarrow$ evals b, $-b$

b: 2x degenerate

evectors for b

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x \\ z \end{pmatrix} \Rightarrow \underset{|b_1\rangle}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}, \underset{|b_2\rangle}{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}}$$

evectors for $-b$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\begin{pmatrix} y \\ x \\ z \end{pmatrix} \Rightarrow \underset{|b_3\rangle}{\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}}$$

e-vals

$$b \begin{cases} |b_1\rangle = |u_3\rangle \\ |b_2\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle) \end{cases}$$

$$-b \; \{ \; |b_3\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle - |u_2\rangle)$$

$$\mathcal{P}(b) = \langle\psi(t)| \left[|b_1\rangle\langle b_1| + |b_2\rangle\langle b_2| \right] |\psi(t)\rangle$$

$$= \langle\psi(t)| \left[|u_3\rangle\langle u_3| + \tfrac{1}{2}(|u_1\rangle + |u_2\rangle)(\langle u_1| + \langle u_2|) \right] |\psi(t)\rangle$$

where $|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega_0 t}|u_1\rangle + \frac{1}{2} e^{-i2\omega_0 t}(|u_2\rangle + |u_3\rangle)$

$$\Rightarrow \mathcal{P}(b) = \frac{1}{2}\left[e^{+i\omega_0 t}\langle u_1| + \frac{1}{\sqrt{2}} e^{+i2\omega_0 t}(\langle u_2| + \langle u_3|) \right]$$

$$\times \left[|u_3\rangle\langle u_3| + \tfrac{1}{2}|u_1\rangle\langle u_1| + \tfrac{1}{2}|u_2\rangle\langle u_2| + \tfrac{1}{2}|u_1\rangle\langle u_2| + \tfrac{1}{2}|u_2\rangle\langle u_1| \right]$$

$$\times \left[e^{-i\omega_0 t}|u_1\rangle + \frac{1}{\sqrt{2}} e^{-i2\omega_0 t}(|u_2\rangle + |u_3\rangle) \right]$$

$$\frac{1}{2}\left[\frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{2\sqrt{2}} e^{-i\omega_0 t} + \frac{1}{2\sqrt{2}} e^{+i\omega_0 t} \right]$$

$$= \frac{5}{8} + \frac{1}{2\sqrt{2}}\left(\frac{1}{2} e^{-i\omega_0 t} + \frac{1}{2} e^{i\omega_0 t} \right)$$

$$= \frac{5}{8} + \frac{1}{2\sqrt{2}} \cos(\omega_0 t)$$

$$\mathcal{P}(-b) = \langle \psi(t) | b_3 \rangle\langle b_3 | \psi(t) \rangle$$

$$= \frac{1}{2}\Big[e^{i\omega_0 t}\langle u_1| + \frac{1}{\sqrt{2}}e^{i2\omega_0 t}\left(\langle u_2| + \langle u_3|\right)$$

$$\times \left[\frac{1}{2}|u_1\rangle\langle u_1| + \frac{1}{2}|u_2\rangle\langle u_2| - \frac{1}{2}|u_1\rangle\langle u_2| - \frac{1}{2}|u_2\rangle\langle u_1|\right]$$

$$\times \left[e^{-i\omega_0 t}|u_1\rangle + \frac{1}{\sqrt{2}}e^{-i2\omega_0 t}\left(|u_2\rangle + |u_3\rangle\right)\right]\Big]$$

$$= \frac{1}{2}\left[\frac{1}{2} + \frac{1}{4} - \frac{1}{2\sqrt{2}}e^{-i\omega_0 t} - \frac{1}{2\sqrt{2}}e^{i\omega_0 t}\right]$$

$$= \frac{3}{8} - \frac{1}{2\sqrt{2}}\cos(\omega_0 t)$$

Check: $\langle B \rangle(t)$

$$= b \cdot \mathcal{P}(b) - b \cdot \mathcal{P}(-b)$$

$$= b\left[\frac{5}{8} + \frac{1}{2\sqrt{2}}\cos(\omega_0 t) - \frac{3}{8} + \frac{1}{2\sqrt{2}}\cos(\omega_0 t)\right.$$

$$= b\left[\frac{1}{4} + \frac{1}{\sqrt{2}}\cos(\omega_0 t)\right]$$

$$= \frac{b}{4} + \frac{b}{\sqrt{2}}\cos(\omega_0 t)$$

Same as in part (c)

V

(a) $[H(t), H(t')] \neq 0$ (given)

$$[H(t), \int_{t_0}^{t} H(t')\,dt']$$
$$= H(t)\int_{t_0}^{t} H(t')\,dt' - \left(\int_{t_0}^{t} H(t')\,dt'\right)H(t)$$
$$= \int_{t_0}^{t} H(t)\,H(t')\,dt' - \int_{t_0}^{t} H(t')\,H(t)\,dt'$$
$$= \int_{t_0}^{t} [H(t), H(t')]\,dt'$$

$\neq 0$ if $[H(t), H(t')] \neq 0$ for arbitrary t, t'

(Similarly, if $[H(t), \int_{t_0}^{t} H(t')\,dt'] \neq 0$ then $[H(t), H(t')] \neq 0$

(b) $F(t) = -\frac{i}{\hbar}\int_{t_0}^{t} H(t')\,dt'$

$$\frac{d}{dt}F(t) = -\frac{i}{\hbar}\frac{d}{dt}\int_{t_0}^{t} H(t')\,dt' = \frac{-i}{\hbar}\lim_{\delta t\to 0}\left[\frac{\int_{t_0}^{t+\delta t} H(t')\,dt' - \int_{t_0}^{t} H(t')\,dt'}{\delta t}\right]$$

(definition of derivative)

$$= \frac{-i}{\hbar}\lim_{\delta t\to 0}\left[\frac{1}{\delta t}\int_{t}^{t+\delta t} H(t')\,dt'\right]$$

$$= \frac{-i}{\hbar}\,\frac{1}{\delta t}\cdot H(t)\cdot \delta t$$

$$\Rightarrow \frac{dF(t)}{dt} = \frac{-i}{\hbar}H(t)$$

$$\Rightarrow \left[F(t), dF(t)/dt\right]$$

$$= \left(\frac{-i}{\hbar}\right)^2 \left[\int_{t_0}^{t} H(t')dt' \, , \; H(t)\right]$$

$$= -\frac{1}{\hbar^2}\left[\int_{t_0}^{t} H(t')dt' \, , \; H(t)\right] \quad \checkmark$$

$$= \frac{1}{\hbar^2}\left[H(t), \; \int_{t_0}^{t} H(t')dt'\right] \quad \checkmark$$

(c) $$\frac{d}{dt}\left(e^{F(t)}\right) = \frac{d}{dt}\left(1 + F(t) + \frac{1}{2}F^2(t) + \frac{1}{3!}F^3(t) + \ldots\right)$$

$$= \frac{dF(t)}{dt} + \frac{1}{2}\left(\frac{dF}{dt}\cdot F + F\cdot\frac{dF}{dt}\right) + \frac{1}{3!}\left(\frac{dF}{dt}\cdot F^2 + F\cdot\frac{dF}{dt}\cdot F + F^2\frac{dF}{dt}\right) + \ldots$$

$$= \frac{dF}{dt} + \frac{1}{2}\left(\frac{dF}{dt}\cdot F + \left[F, \frac{dF}{dt}\right] + \frac{dF}{dt}\cdot F\right) + \frac{1}{3!}\left(\cdots\right) + \ldots$$

$$= \frac{dF}{dt}\underbrace{\left(1 + F(t) + \frac{1}{2}\left[F, dF/dt\right] + \text{Higher order terms}\right)}_{\text{Only } = \text{ to } e^{F(t)} \text{ if } [F,\, dF/dt] = 0}$$

$\rightarrow$ if $[F, dF/dt] \neq 0$, then $\frac{d}{dt}e^{F(t)} \neq \frac{dF}{dt}\cdot e^{F(t)}$

if $[F, dF/dt] = 0$, then $\frac{d}{dt}e^{F(t)} = \frac{dF}{dt}e^{F(t)}$

This could have been done more rigorously, but for the goal of illustrating the main idea, this suffices. But see next page for alternate solution.

Alternative solution

$$\frac{d}{dt} e^{F(t)} = \frac{d}{dt} F(t) + \frac{1}{2!}\frac{d}{dt} F^2(t) + \frac{1}{3!}\frac{d}{dt} F^3(t) + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!}\frac{d}{dt}\left(F^n(t)\right)$$

$$\frac{dF}{dt} e^{F(t)} = \frac{dF}{dt} \sum_{m=0}^{\infty} \frac{1}{m!} F^m(t)$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{dF}{dt} \cdot F^m(t) = \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \frac{dF}{dt} F^{(m-1)}(t)$$

$\Rightarrow$ for any n, need $\displaystyle \frac{1}{n!}\frac{d}{dt}(F^n(t)) = \frac{1}{(n-1)!}\frac{dF}{dt} F^{(n-1)}(t)$

$$\Rightarrow \frac{d}{dt} F^n(t) = n \frac{dF}{dt} F^{(n-1)}(t)$$

if this is true, $\displaystyle \frac{d}{dt} e^{F(t)} = \frac{dF}{dt} e^{F(t)}$

if not true, $\neq$

$\underline{n=1}$

$$\frac{dF}{dt} = \frac{dF}{dt} \quad \checkmark$$

$\underline{n=2}$

$$\frac{d}{dt}F^2 = \frac{d}{dt}(F \cdot F) = \frac{dF}{dt}\cdot F + F\frac{dF}{dt} = \frac{dF}{dt}F + [F, dF/dt] + \frac{dF}{dt} = 2\frac{dF}{dt}F + [F, \frac{dF}{dt}]$$

$\Rightarrow \; = 2\frac{dF}{dt}F$ only if $[F, dF/dt] = 0$

$\Rightarrow$ IF $[F, dF/dt] \neq 0$, then $\displaystyle \frac{d}{dt}e^{F(t)} \neq \frac{dF}{dt} e^{F(t)}$

(d) Assume $U(t,t_0) = e^{-\frac{i}{\hbar}\int_{t_0}^{t} H(t')dt'} = e^{F(t)}$

using previous definition of $F(t)$

Substitute into evolution equation:

$i\hbar \frac{d}{dt} U(t,t_0) = i\hbar \frac{d}{dt} e^{F(t)} = H(t) e^{F(t)}$ (using eq. 4, p.308)

but $H(t) = i\hbar \frac{d}{dt} F(t)$ (from part (b))

$\Rightarrow i\hbar \frac{d}{dt} e^{F(t)} = i\hbar \frac{dF}{dt} e^{F(t)}$

$\Rightarrow \frac{d}{dt} e^{F(t)} = \frac{dF}{dt} e^{F(t)}$

From part (c), we know this is true if and only if

$[F, dF/dt] = 0$

which is true if and only if

$[H(t), \int_{t_0}^{t} H(t')dt'] = 0$ (part b)

which is true if and only if $[H(t), H(t')] = 0$
(part a)

So $\boxed{U(t,t_0) = e^{-\frac{i}{\hbar}\int_{t_0}^{t} H(t')dt'} \quad \text{IF and ONLY IF} \quad [H(t), H(t')] = 0}$

OPTIONAL PROBLEMS

VI

$\boxed{\text{CT III \#4}}$

(a) Since this is a free particle problem, $dV/dx = 0$

$$\Rightarrow \frac{d}{dt}\langle P\rangle = 0 \quad \Rightarrow \quad \langle P\rangle = \text{constant in time}$$

$$\langle P\rangle(t) = \langle P\rangle_0$$

$\leftarrow \langle P\rangle$ at time $t=0$

$$\frac{d}{dt}\langle x\rangle = \frac{1}{m}\langle P\rangle = \frac{1}{m}\langle P\rangle_0$$

So $\langle x\rangle(t) = \frac{1}{m}\langle P\rangle_0 t + \langle x\rangle_0$

$\leftarrow \langle x\rangle$ at $t=0$

So $\langle x\rangle$ is a linear function of time

(b) $$\frac{d}{dt}\langle x^2\rangle = \frac{1}{i\hbar}\langle [x^2, H]\rangle + \left\langle \cancel{\frac{\partial x^2}{\partial t}}\right\rangle^{0} = \frac{1}{i\hbar}\langle [x^2, H]\rangle$$

$$= \frac{1}{i\hbar}\langle [x^2, P^2/2m]\rangle$$

Now,
$$\begin{aligned}
[x^2, P^2] &= xxPP - PPxx \\
&= x([x,P] + Px)P - PPxx \\
&= i\hbar xP + xPxP - PPxx \\
&= i\hbar xP + (i\hbar + Px)xP - PPxx \\
&= i\hbar xP + i\hbar xP + PxxP - PPxx \\
&= 2i\hbar xP + P(xxP - Pxx) \\
&= 2i\hbar xP + P[x^2, P] \\
&= 2i\hbar xP - P[P, x^2] \\
&= 2i\hbar xP - P[P,x]\cdot 2x \\
&= 2i\hbar xP + 2i\hbar Px \\
&= 2i\hbar(xP + Px)
\end{aligned}$$

$$\Rightarrow \frac{d}{dt}\langle x^2\rangle = \frac{1}{i\hbar}\langle 2i\hbar(xP+Px)/2m\rangle = \frac{1}{m}\langle xP + Px\rangle$$

Also, $$\frac{d}{dt}\langle xP + Px\rangle = \frac{1}{i\hbar}\langle [xP+Px, P^2/2m]\rangle = \frac{1}{2mi\hbar}\langle [xP, P^2] + [Px, P^2]\rangle$$

$$= \frac{1}{2mi\hbar}\langle (xP^2 - P^2x)P + P(xP^2 - P^2x)\rangle$$

$$= \frac{1}{2mi\hbar}\langle [x,P^2]P + P[x,P^2]\rangle$$

$$= \frac{1}{2mi\hbar}\left\langle [x,p](2p)p + p[x,p](2p) \right\rangle$$

$$= \frac{1}{m}\left\langle p^2 + p^2 \right\rangle$$

$$\boxed{\begin{aligned} \frac{d}{dt}\langle xp+px \rangle &= \frac{2}{m}\langle p^2 \rangle \\ \frac{d}{dt}\langle x^2 \rangle &= \frac{1}{m}\langle xp+px \rangle \end{aligned}}$$

To integrate, we need to also know the time-dependence of $\langle p^2 \rangle$.

$$\frac{d}{dt}\langle p^2 \rangle = \frac{1}{i\hbar}\left\langle [p^2, p^2/2m] \right\rangle = 0 \Rightarrow \langle p^2 \rangle(t) = \langle p^2 \rangle_0$$

$$\Rightarrow \frac{d}{dt}\langle xp+px \rangle = \frac{2\langle p^2 \rangle_0}{m} \Rightarrow \langle xp+px \rangle(t) = \frac{2\langle p^2 \rangle_0 t}{m} + \langle xp+px \rangle_0$$

$$\Rightarrow \frac{d}{dt}\langle x^2 \rangle = \frac{1}{m}\cdot\left(\frac{2\langle p^2 \rangle_0 t}{m} + \langle xp+px \rangle_0\right)$$

integrate : $\boxed{\langle x^2 \rangle(t) = \frac{\langle p^2 \rangle_0 t^2}{m^2} + \frac{\langle xp+px \rangle_0 t}{m} + \langle x^2 \rangle_0}$

(c) $$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$= \frac{\langle p^2 \rangle_0 t^2}{m^2} + \frac{\langle xp+px \rangle_0 t}{m} + \langle x^2 \rangle_0 - \left(\frac{1}{m}\langle p \rangle_0 t + \langle x \rangle_0\right)^2$$

$$= \frac{\langle p^2 \rangle_0 t^2}{m^2} - \frac{\langle p \rangle_0^2 t^2}{m^2} + \left(\frac{\langle xp+px \rangle_0}{m} - \frac{2\langle p \rangle_0 \langle x \rangle_0}{m}\right)t + \langle x^2 \rangle_0 - \langle x \rangle_0^2$$

$$= \frac{(\Delta p)_0^2 t^2}{m^2} + (\Delta x)_0^2 + \left(\frac{\langle xp+px \rangle_0}{m} - \frac{2\langle p \rangle_0 \langle x \rangle_0}{m}\right)t$$

Rewrite this:

$$(\Delta x)^2(t) = at^2 + b_0 t + c_0$$

where $a = (\Delta p)_0^2/m^2$

$$b = \frac{1}{m}\left(\langle xp + px\rangle_0 - 2\langle p\rangle_0\langle x\rangle_0\right)$$

$$c = (\Delta x)_0^2$$

Want: let $t' = t + \delta t$ so that

$$(\Delta x)^2(t') = \frac{1}{m^2}(\Delta p)^2_{t'=0}(t')^2 + (\Delta x)^2_{t'=0}$$

$$\Rightarrow (\Delta x)^2(t) = at^2 + b_0 t + c_0$$

$$= a\left(t^2 + \frac{b_0}{a}t\right) + c_0$$

$$= a\left(t^2 + \frac{b_0}{a}t + \frac{b_0^2}{4a^2}\right) + c_0 - \frac{b_0^2}{4a}$$

$$= a\left(t + \frac{b_0}{2a}\right)^2 + \left(c_0 - \frac{b_0^2}{4a}\right)$$

let $t' = t + \frac{b_0}{2a} \Rightarrow \delta t = \frac{b_0}{2a}$

$$\Rightarrow (\Delta x)^2(t') = at'^2 + \left(c_0 - b_0^2/4a\right)$$

$$= \frac{(\Delta p)^2_{t=0}}{m^2}t'^2 + \left(c_{t=0} - b^2_{t=0}/4a\right)$$

since Δp is const in time, $(\Delta p)_{t=0} = (\Delta p)_{t'=0}$

$$\Rightarrow (\Delta x)^2(t') = \frac{(\Delta p)^2_{t'=0}}{m^2}(t')^2 + \left(c_{t=0} - b^2_{t=0}/4a\right)$$

For $t' = 0$, the above equation gives

$$(\Delta x)^2(t'=0) = C_{t=0} - \frac{b_{t=0}^2}{4a}$$

So we can write

$$(\Delta x)^2(t') = \frac{(\Delta p)^2_{t'=0}}{m^2} \cdot t'^2 + (\Delta x)^2_{t'=0}$$

where $t' = t + \frac{b_0}{2a}$ means that we have shifted the time origin by an amount $\delta t = \frac{b_0}{2a}$ (Since $t' = \frac{b_0}{2a}$ for $t=0$, then we have shifted the time origin back by $b_0/2a$)

Let's figure out what this shift is, in terms of quantities we have calculated.

$$\delta t = \frac{m^2}{2(\Delta p)_0^2} \cdot \frac{1}{m}\left(\langle xp + px \rangle_0 - 2\langle p \rangle_0 \langle x \rangle_0 \right)$$

$$= \frac{m}{2(\Delta p)_0^2}\left(\langle xp + px \rangle_{t=0} - 2\langle p \rangle_{t=0} \langle x \rangle_{t=0} \right)$$

$\Rightarrow$ by shifting the time origin back this amount, we can write the expression for the width of the wavepacket in a simpler form.

Interpretation:

The width of the wavepacket increases quadratically with time:

$$(\Delta x)(t') = \left(\frac{(\Delta p)_0^2}{m^2} t'^2 + (\Delta x)_{t'=0} \right)^{1/2}$$

With a larger initial Δp (ie, a wider range of momentum components), Δx increases faster).

Note also the similarity between $(\Delta x)(t')$ and the width of a gaussian laser beam propagating along the z-axis:

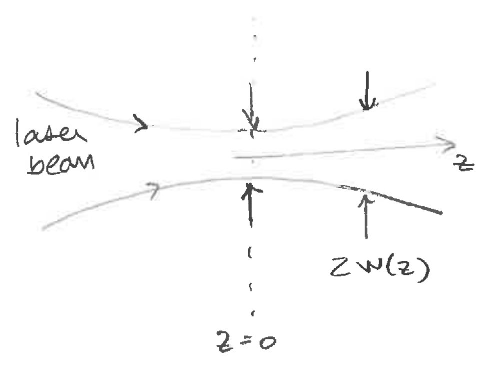
$$w(z) = w(0) \cdot \left(1 + z^2/z_0^2\right)^{1/2} \quad \text{where} \quad z_0 = \pi\, w(0)^2/\lambda$$

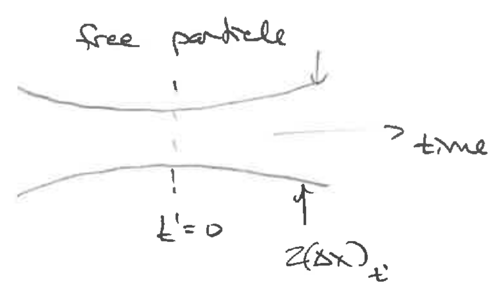
wavelength

In our problem,

$$(\Delta x)_{t'} = (\Delta x)_{t'=0} \left(1 + t'^2/\tau^2\right)^{1/2}$$

$$\text{where} \quad \tau = \frac{m\,(\Delta x)_{t'=0}}{(\Delta p)_{t'=0}}$$



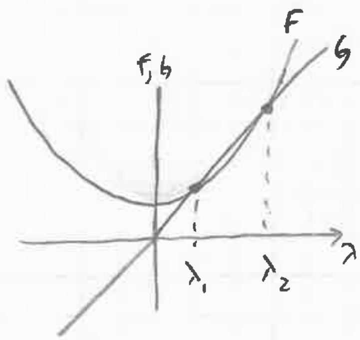


VII. PART A

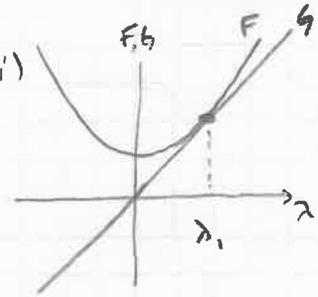
$F(\lambda) = a\lambda^2 + c \qquad a, c \in \mathbb{R}, > 0$

$G(\lambda) = b\lambda \qquad b \in \mathbb{R}$

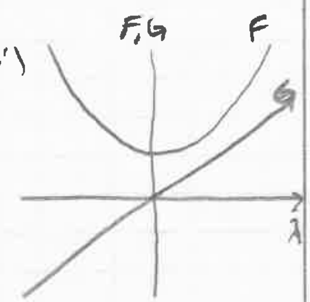
(a)



(ii)



(iii)



(b) $F(\lambda) = G(\lambda)$

$a\lambda^2 + c = b\lambda \Rightarrow a\lambda^2 - b\lambda + c = 0$

$\Rightarrow \lambda = \dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$

$D = b^2 - 4ac > 0$ in order for there to be 2 real solutions.

(c) $\boxed{D \leq 0}$ implies $F(\lambda) \geq G(\lambda)$ for all λ

VII | PART B

(a) $F = F^+$

$$G = iF \Rightarrow G^+ = -iF^+ = -iF = -G$$

$$\Rightarrow G = -G^+ \,, \quad G \text{ is anti-Hermitian}$$

(b) Let $G = iF$ where F is Hermitian

Let $F|u_n\rangle = f_n |u_n\rangle$, $f_n \in \mathbb{R}$

$$G|u_n\rangle = iF|u_n\rangle = (if_n)|u_n\rangle$$

$\hookrightarrow$ eigenvalues of G are if_n, imag. #s

$$\langle G\rangle = \langle iF\rangle = i\langle F\rangle$$

F Hermitian $\Rightarrow \langle F\rangle \in \mathbb{R} \Rightarrow \langle G\rangle$ imaginary

(c) $[A,B]^+ = (AB - BA)^+ = (AB)^+ - (BA)^+ = B^+A^+ - A^+B^+$

$= [B^+, A^+] = [B, A] = -[A,B]$.

$\Rightarrow [A,B] = -[A,B]^+ \Rightarrow [A,B]$ Anti-Hermitian

$\rightarrow \langle [A,B]\rangle$ is a purely imaginary #.

VII PART C

$[A,B] = iC \Rightarrow C = -i(AB - BA)$

$|\varphi\rangle = (A + i\lambda B)|\psi\rangle$ which holds for any $\lambda \in \mathbb{R}$

(a) $\langle\varphi|\varphi\rangle = \langle\psi|(A^{+} - i\lambda B^{+})(A + i\lambda B)|\psi\rangle$

$= \langle\psi|(A - i\lambda B)(A + i\lambda B)|\psi\rangle$ since A, B observables

$= \langle\psi|A^2 + \lambda^2 B^2 + i\lambda AB - i\lambda BA|\psi\rangle$

$= \langle A^2\rangle + \lambda^2\langle B^2\rangle + i\lambda\langle AB - BA\rangle$

$= \langle A^2\rangle + \lambda^2\langle B^2\rangle - \lambda\langle C\rangle$

Since $\langle\varphi|\varphi\rangle \geq 0$ (scalar product)

$\lambda^2\langle B^2\rangle - \lambda\langle C\rangle + \langle A^2\rangle \geq 0$ $\rightarrow \lambda^2\langle B^2\rangle + \langle A^2\rangle \geq \lambda\langle C\rangle$ for all λ

$D = \langle C\rangle^2 - 4\langle A^2\rangle\langle B^2\rangle \leq 0$

$\Rightarrow 4\langle A^2\rangle\langle B^2\rangle \geq \langle C\rangle^2$

$\Rightarrow \langle A^2\rangle\langle B^2\rangle \geq \frac{1}{4}\langle -i[A,B]\rangle^2$

since $\langle[A,B]\rangle$ must be purely imaginary,

$i[A,B] \in \mathbb{R}$

$\downarrow$

$\langle A^2\rangle\langle B^2\rangle \geq \frac{1}{4}\left|\langle[A,B]\rangle\right|^2$

(b) $A' = A - \langle A\rangle$, $B' = B - \langle B\rangle$

$[A', B'] = [A - \langle A\rangle, B - \langle B\rangle] = [A, B - \langle B\rangle] - \cancel{[\langle A\rangle, B - \langle B\rangle]}^{\,0}$

$= [A,B] - \cancel{[A, \langle B\rangle]}^{\,0}$

$= [A,B]$ since $\langle A\rangle, \langle B\rangle$ are #s

$\Rightarrow \langle(A')^2\rangle\langle(B')^2\rangle \geq \frac{1}{4}\left|\langle[A,B]\rangle\right|^2$

(c) $(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2$, $(\Delta B)^2 = \langle B^2\rangle - \langle B\rangle^2$

Consider $(A')^2$

$$(A')^2 = (A - \langle A\rangle)^2$$
$$= A^2 + \langle A\rangle^2 - 2A\langle A\rangle$$
$$\Rightarrow \langle (A')^2\rangle = \langle A^2\rangle + \langle A\rangle^2 - 2\langle A\rangle^2$$
$$= \langle A^2\rangle - \langle A\rangle^2 = (\Delta A)^2$$

Similarly, $\langle (B')^2\rangle = (\Delta B)^2$

$$\Rightarrow (\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\left|\langle [A,B]\rangle\right|^2$$

(d) $[Q, P] = i\hbar$

$$\Rightarrow (\Delta Q)^2(\Delta P)^2 \geq \frac{1}{4}|i\hbar|^2$$

$$(\Delta Q)^2(\Delta P)^2 \geq \frac{\hbar^2}{4}$$

$$\downarrow$$

$$\Delta Q\,\Delta P \geq \frac{\hbar}{2}$$

($\Delta Q \Delta P \geq -\hbar/2$ doesn't have physical relevance since $\Delta Q \geq 0$ and $\Delta P \geq 0$ for <u>ANY</u> Q, P that are observables)