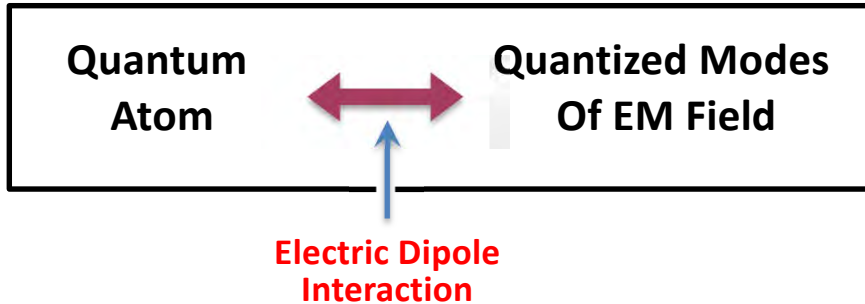


Quantized Light – Matter Interactions

General Problem:



Starting Point: System Hamiltonian

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} \quad (1)$$

$$\hat{H}_F = \sum_{\vec{k}} \hbar \omega_{\vec{k}} (\hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} + \frac{1}{2}) \quad \text{Field}$$

$$\hat{H}_A = \sum_i E_i |i\rangle \langle i| = \sum_i E_i \hat{\sigma}_{ii} \quad \text{Atom}$$

$$\hat{H}_{AF} = -\hat{\vec{p}} \cdot \hat{\vec{E}}(\vec{r}, t) \quad \text{ED interaction}$$

$E_i, |i\rangle$: energies, energy levels of the atom

Dipole Operator:

$$(2) \quad \hat{\vec{p}} = \sum_{i,j} \vec{p}_{ij} |i\rangle \langle j| = \sum_{i,j} \vec{p}_{ij} \hat{\sigma}_{ij}$$

Field Operator:

$$\hat{\vec{E}}(\vec{r}, t) = \sum_{\vec{k}} \vec{\epsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + \text{H.c.}, \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar \omega_{\vec{k}}}{2 \epsilon_0 V}}$$

2 polarization modes implicit

Pin down atom where $u_{\vec{k}}(\vec{r}) = 1$

– anywhere if $u_{\vec{k}}(\vec{r}) = e^{i\vec{k} \cdot \vec{r}}$

– if $u_{\vec{k}}(\vec{r}) = \sin(kz)$ then where $\sin(kz) = 1$



$$(3) \quad \hat{\vec{E}}(\vec{r}, t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\epsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} (\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^\dagger)$$

Quantized Light – Matter Interactions

Dipole Operator:

(2)
$$\hat{\vec{p}} = \sum_{i,j} \vec{p}_{ij} |i\rangle\langle j| = \sum_{i,j} \vec{p}_{ij} \hat{\sigma}_{ij}$$

Field Operator:

$$\vec{E}(\vec{r}, t) = \sum_{\vec{k}} \vec{\epsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + \text{H.c.}, \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar \omega_{\vec{k}}}{2 \epsilon_0 V}}$$

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(3)
$$\vec{E}(\vec{r}, t) = \vec{E}(t) = \sum_{\vec{k}} \vec{\epsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} (\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^{\dagger})$$

Combining (2) & (3):

$$\begin{aligned} \hat{H}_{AF} &= \sum_{i,j} \sum_{\vec{k}} -\vec{p}_{ij} \cdot \vec{\epsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^{\dagger}) \\ &= \sum_{i,j} \sum_{\vec{k}} \hbar g_{\vec{k}}^{(ij)} \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^{\dagger}) \end{aligned}$$

where $g_{\vec{k}}^{(ij)} = \frac{\vec{p}_{ij} \cdot \vec{\epsilon}_{\vec{k}} \mathcal{E}_{\vec{k}}}{\hbar}$

Rabi Freq., note sign convention

2-level atom $\Rightarrow (i,j) = (1,2)$:

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} (g_{\vec{k}} \hat{\sigma}_{21} + g_{\vec{k}}^* \hat{\sigma}_{12}) (\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^{\dagger})$$

Define:

$$\hat{\sigma}_+ = \hat{\sigma}_{21} = |2\rangle\langle 1|$$

$$\hat{\sigma}_- = \hat{\sigma}_{12} = |1\rangle\langle 2|$$

$$\hat{\sigma}_z = \hat{\sigma}_{22} - \hat{\sigma}_{11} = |2\rangle\langle 2| - |1\rangle\langle 1|$$

Pauli matrices

$$\hat{\sigma}_x = \frac{1}{2}(\hat{\sigma}_+ + \hat{\sigma}_-)$$

$$\hat{\sigma}_y = \frac{1}{2i}(\hat{\sigma}_+ - \hat{\sigma}_-)$$

$$\hat{\sigma}_z$$

Quantized Light – Matter Interactions

Combining (2) & (3):

$$\hat{H}_{AF} = \sum_{ij} \sum_{\vec{k}} -\vec{p}_{ij} \cdot \vec{\epsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^{\dagger})$$

$$= \sum_{ij} \sum_{\vec{k}} \hbar g_{\vec{k}}^{(ij)} \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^{\dagger})$$

where $g_{\vec{k}}^{(ij)} = \frac{\vec{p}_{ij} \cdot \vec{\epsilon}_{\vec{k}} \mathcal{E}_{\vec{k}}}{\hbar}$

Rabi Freq., note sign convention

2-level atom $\Rightarrow (i,j) = (1,2) :$

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} (g_{\vec{k}} \hat{\sigma}_{21} + g_{\vec{k}}^* \hat{\sigma}_{12}) (\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^{\dagger})$$

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$$\hat{\sigma}_x = \frac{1}{2} (\hat{\sigma}_+ + \hat{\sigma}_-)$$

$$\hat{\sigma}_y = \frac{1}{2i} (\hat{\sigma}_+ - \hat{\sigma}_-)$$

$$\hat{\sigma}_z$$

With this notation

(4)

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} (g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + \cancel{g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}}^{\dagger}} + \cancel{g_{\vec{k}} \hat{\sigma}_- \hat{a}_{\vec{k}}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}_{\vec{k}}^{\dagger})$$

Energy conservation?



$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} (g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}_{\vec{k}}^{\dagger})$$

Putting it all together

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} =$$

(5)

$$\sum_{\vec{k}} \hbar \omega_{\vec{k}} \hat{a}_{\vec{k}}^{\dagger} \hat{a}_{\vec{k}} + \frac{1}{2} \hbar \omega_{21} \hat{\sigma}_z + \sum_{\vec{k}} \hbar (g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}_{\vec{k}}^{\dagger})$$

We changed the zero point for energy by subtracting

$$\sum_{\vec{k}} \frac{1}{2} \hbar \omega_{\vec{k}} \quad \text{field} \quad \text{and} \quad \frac{1}{2} (E_2 - E_1) \quad \text{atom}$$

Quantized Light – Matter Interactions

With this notation

(4)

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} (g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + \cancel{g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}}^+} + \cancel{g_{-\vec{k}} \hat{\sigma}_- \hat{a}_{-\vec{k}}} + g_{-\vec{k}} \hat{\sigma}_- \hat{a}_{-\vec{k}}^+)$$

Energy conservation?



$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} (g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{-\vec{k}} \hat{\sigma}_- \hat{a}_{-\vec{k}}^+)$$

Putting it all together

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} = \quad (5)$$

$$\sum_{\vec{k}} \hbar \omega_{\vec{k}} \hat{a}_{\vec{k}}^+ \hat{a}_{\vec{k}} + \frac{1}{2} \hbar \omega_{21} \hat{\sigma}_z + \sum_{\vec{k}} \hbar (g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{-\vec{k}} \hat{\sigma}_- \hat{a}_{-\vec{k}}^+)$$

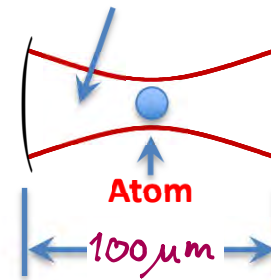
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$$\sum_{\vec{k}} \frac{1}{2} \hbar \omega_{\vec{k}} \quad \text{and} \quad \frac{1}{2} (E_2 - E_1) \quad \begin{array}{c} \text{field} \\ \text{atom} \end{array}$$

Interaction with Single-mode Fields

Good approx. in small, high-Q Cavity

Gaussian beam mode



$$c/2L \gg A_{21}$$

$$|g_{\vec{k}}| \gg A_{21} \delta$$

Single-mode (Jaynes-Cummings) Hamiltonian

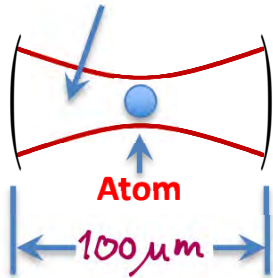
$$\hat{H} = \underbrace{\hbar \omega \hat{a}^+ \hat{a}}_{H_0} + \underbrace{\frac{1}{2} \hbar \omega_{21} \hat{\sigma}_z + \hbar g (\hat{\sigma}_+ + \hat{\sigma}_-) (\hat{a} + \hat{a}^+)}_{H_{AF}}$$

Quantized Light – Matter Interactions

Interaction with Single-mode Fields

Good approx. in small, high-Q Cavity

Gaussian beam mode



$$c/2L \gg A_{21}$$

$$|g_{\vec{k}}| \gg A_{21}, \gamma$$

Single-mode (Jaynes-Cummings) Hamiltonian

$$\hat{H} = \underbrace{\hbar\omega \hat{a}^\dagger \hat{a} + \frac{1}{2}\hbar\omega_2 \hat{\sigma}_z}_{H_0} + \underbrace{\hbar g(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger)}_{H_{AF}}$$

Foundational result for
remainder of the course

Quantized Light – Matter Interactions

Paradigm for spin-1/2 coupled to QHO

- Atom in high-Q Cavity ***)**
- Quantum dot in high-Q Cavity
- Rydberg atom in superconducting μw Cavity
- Superconducting qubit in superconducting μw Cavity
- Superconducting qubit in superconducting μw stripline Cavity (circuit QED)
- Trapped ion with quantized COM motion ***)**

***) Nobel Prize in Physics 2012**

More about Single-mode Cavity QED

$$\hat{H} = \underbrace{\hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\omega_2\hat{\sigma}_z}_{H_0} + \underbrace{\hbar(g_{\vec{R}}\hat{\sigma}_+ + g_{\vec{R}}^*\hat{\sigma}_-)(\hat{a}_{\vec{R}} + \hat{a}_{\vec{R}}^\dagger)}_{H_{AF}}$$

For simplicity $\vec{p}_{21} = \vec{p}_{12} \Rightarrow g_{\vec{R}} = g_{\vec{R}}^*$

Note: $\hat{H}_{AF}$ conserves excitation number,
couples $|2, n\rangle \leftrightarrow |1, n+1\rangle$



Series of 2-level systems, one for each n

All 2-level systems are alike
Rabi problem!

Switch to Interaction Picture:

**Sakurai page
318-319**

$$\begin{aligned}\hat{H}_S &\rightarrow \hat{H}_I = e^{i\frac{\hat{H}_0}{\hbar}t} \hat{H}_{AF} e^{-i\frac{\hat{H}_0}{\hbar}t} \\ |\psi_S(t)\rangle &\rightarrow |\psi_I(t)\rangle = e^{i\frac{\hat{H}_0}{\hbar}t} |\psi_S(t)\rangle\end{aligned}$$



Quantized Light – Matter Interactions

More about Single-mode Cavity QED

$$\hat{H} = \underbrace{\hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\omega_2\hat{\sigma}_2}_{H_0} + \underbrace{\hbar(g_R\hat{\sigma}_+ + g_R^*\hat{\sigma}_-)(\hat{a}_R + \hat{a}_R^\dagger)}_{H_{AF}}$$

For simplicity $\vec{n}_{21} = \vec{n}_{12} \Rightarrow g_R = g_R^*$

Note: $\hat{H}_{AF}$ conserves excitation number,
couples $|2, n\rangle \leftrightarrow |1, n+1\rangle$



Series of 2-level systems, one for each n

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**Sakurai page
318-319**

$$\left. \begin{aligned} \hat{H}_S &\rightarrow \hat{H}_I = e^{i\frac{\hat{H}_0}{\hbar}t} \hat{H}_{AF} e^{-i\frac{\hat{H}_0}{\hbar}t} \\ |\psi_S(t)\rangle &\rightarrow |\psi_I(t)\rangle = e^{i\frac{\hat{H}_0}{\hbar}t} |\psi_S(t)\rangle \end{aligned} \right\}$$



$$\hat{H}_{AF} = \hbar g (\hat{\sigma}_+ \hat{a} e^{i(\omega_2 - \omega)t} + \hat{\sigma}_+ \hat{a}^\dagger e^{i(\omega_2 + \omega)t} + \hat{\sigma}_- \hat{a} e^{-i(\omega_2 + \omega)t} + \hat{\sigma}_- \hat{a}^\dagger e^{-i(\omega_2 - \omega)t})$$

Can
show

$$\begin{aligned} e^{i\omega\hat{a}^\dagger\hat{a}t} \hat{a} e^{-i\omega\hat{a}^\dagger\hat{a}t} &= \hat{a} e^{-i\omega t} \\ e^{i\frac{\omega_2}{2}\hat{\sigma}_2t} \hat{\sigma}_+ e^{-i\frac{\omega_2}{2}\hat{\sigma}_2t} &= \hat{\sigma}_+ e^{-i\omega_2t} \end{aligned}$$

Quantized Light – Matter Interactions

More about Single-mode Cavity QED

$$\hat{H} = \underbrace{\hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar(g_R\hat{\sigma}_+ + g_R^*\hat{\sigma}_-)(\hat{a}_R + \hat{a}_R^\dagger)}_{H_{AF}}$$

For simplicity $\vec{p}_{21} = \vec{p}_{12} \Rightarrow g_R = g_R^*$

Note: $\hat{H}_{AF}$ conserves excitation number, couples $|2, n\rangle \leftrightarrow |1, n+1\rangle$

Series of 2-level systems, one for each n

All 2-level systems are alike
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Sakurai page
318-319

$$\left. \begin{aligned} \hat{H}_S &\rightarrow \hat{H}_I = e^{i\frac{\hat{H}_0}{\hbar}t} \hat{H}_{AF} e^{-i\frac{\hat{H}_0}{\hbar}t} \\ |\psi_S(t)\rangle &\rightarrow |\psi_I(t)\rangle = e^{i\frac{\hat{H}_0}{\hbar}t} |\psi_S(t)\rangle \end{aligned} \right\}$$

$$\hat{H}_{AF} = \hbar g (\hat{\sigma}_+ \hat{a} e^{i(\omega_{21}-\omega)t} + \hat{\sigma}_+ \hat{a}^\dagger e^{i(\omega_{21}+\omega)t} + \hat{\sigma}_- \hat{a} e^{-i(\omega_{21}+\omega)t} + \hat{\sigma}_- \hat{a}^\dagger e^{-i(\omega_{21}-\omega)t})$$

RWA and resonant approximation

Jaynes-Cummings Hamiltonian

$$\hat{H}_I = \hat{H}_{AF} = \hbar g (\hat{\sigma}_+ \hat{a} e^{i\Delta t} + \hat{\sigma}_- \hat{a}^\dagger e^{-i\Delta t})$$

$\Delta = \omega_{21} - \omega$

Important: Change in notation

From this point on we change indices **1** -> **g** (ground state) and **2** -> **e** (excited state). Thus, when a **g** appears in a ket or as a subscript to a probability amplitude, it refers to a state. Elsewhere **g** is a vacuum Rabi frequency.

Eigenstates of	$\hat{H}_0 = \hat{H}_F + \hat{H}_A$
State	Energy
$ e, n\rangle$	$\hbar\omega n + \frac{1}{2}\hbar\omega_{21}$
$ g, n+1\rangle$	$\hbar\omega(n+1) - \frac{1}{2}\hbar\omega_{21}$

Quantized Light – Matter Interactions

$$\hat{H}_{AF} = \hbar g (\hat{\sigma}_+ \hat{a} e^{i(\omega_2 - \omega)t} + \hat{\sigma}_+ \hat{a}^\dagger e^{i(\omega_2 + \omega)t} + \hat{\sigma}_- \hat{a} e^{-i(\omega_2 + \omega)t} + \hat{\sigma}_- \hat{a}^\dagger e^{-i(\omega_2 - \omega)t})$$

RWA and resonant approximation



Jaynes-Cummings Hamiltonian

$$\hat{H}_I = \hat{H}_{AF} = \hbar g (\hat{\sigma}_+ \hat{a} e^{i\Delta t} + \hat{\sigma}_- \hat{a}^\dagger e^{-i\Delta t})$$

$\Delta = \omega_2 - \omega$

Important: Change in notation

For the remainder of this course we change indices **1** to **g** (ground state) and **2** to **e** (excited state). Thus a **g** appearing inside a ket refers to a state, elsewhere **g** is a Rabi frequency. This is needed for clarity.

Eigenstates of	$\hat{H}_0 = \hat{H}_F + \hat{H}_A$
State	Energy
$ e, n\rangle$	$\hbar\omega n + \frac{1}{2}\hbar\omega_2$
$ g, n+1\rangle$	$\hbar\omega(n+1) - \frac{1}{2}\hbar\omega_2$

Cavity QED version of the Rabi Problem

$$|\psi(0)\rangle = |e, n\rangle$$

$$|\psi(t)\rangle = c_{g, n+1} |g, n+1\rangle + c_{e, n} |e, n\rangle$$

Matrix elements

$$\langle e, n | \hat{H}_{AF} | g, n+1 \rangle = \hbar g \sqrt{n+1} e^{i\Delta t}$$

$$\langle g, n+1 | \hat{H}_{AF} | e, n \rangle = \hbar g \sqrt{n+1} e^{-i\Delta t}$$



Schrödinger Equation

$$i\hbar \frac{d}{dt} \begin{pmatrix} c_{g, n+1} \\ c_{e, n} \end{pmatrix} =$$

$$\hbar g \sqrt{n+1} \begin{pmatrix} 0 & e^{-i\Delta t} \\ e^{i\Delta t} & 0 \end{pmatrix} \begin{pmatrix} c_{g, n+1} \\ c_{e, n} \end{pmatrix}$$

Quantized Light – Matter Interactions

Cavity QED version of the Rabi Problem

$$|\psi(0)\rangle = |e, n\rangle$$

$$|\psi(t)\rangle = c_{g, n+1} |g, n+1\rangle + c_{e, n} |e, n\rangle$$

Matrix elements

$$\langle e, n | \hat{H}_{AF} | g, n+1 \rangle = \hbar g \sqrt{n+1} e^{i\Delta t}$$

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Schrödinger Equation

$$i\hbar \frac{d}{dt} \begin{pmatrix} c_{g, n+1} \\ c_{e, n} \end{pmatrix} =$$

$$\hbar g \sqrt{n+1} \begin{pmatrix} 0 & e^{-i\Delta t} \\ e^{i\Delta t} & 0 \end{pmatrix} \begin{pmatrix} c_{g, n+1} \\ c_{e, n} \end{pmatrix}$$



$$\dot{c}_{g, n+1} = -ig \sqrt{n+1} e^{-i\Delta t} c_{e, n}$$

$$\dot{c}_{e, n} = -ig \sqrt{n+1} e^{i\Delta t} c_{g, n+1}$$

Substitute $c_{g, n+1} \rightarrow c_g$, $c_{e, n} \rightarrow c_e e^{i\Delta t}$

Looks **exactly** like Semiclassical Rabi problem

Solve for $c_g(0) = 0$, $c_e(0) = 1$



$$c_{e, n}(t) = \left[\cos\left(\frac{\Omega_n t}{2}\right) - i \frac{\Delta}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) \right] e^{i\Delta t/2}$$

$$c_{g, n+1} = -i \frac{2g\sqrt{n+1}}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) e^{-i\Delta t/2}$$

$$\Omega_n = (4g^2(n+1) + \Delta^2)^{1/2}$$

Quantized Light – Matter Interactions

Vacuum Rabi Oscillations

If $|\psi(0)\rangle = |e, 0\rangle \Rightarrow$ no photons in field

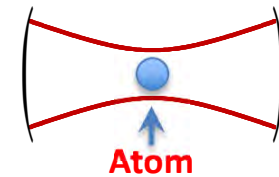
yet $|e, 0\rangle$ evolves into $|g, 1\rangle$

Uniquely QED phenomenon!

Asymmetry $\left\{ \begin{array}{l} |e, n=0\rangle \rightarrow |g, n=1\rangle \\ |g, n=0\rangle \rightarrow |g, n=1\rangle \end{array} \right.$

holds germ of **Spontaneous Decay!**

Next: More Cavity QED



2-level atom

Single cavity mode

What happens with a Coherent State in the Cavity mode?

(Quantum-Classical correspondence)

Initial atom-field state: $\sum_n C_n |g, \alpha\rangle$

$$|\psi(0)\rangle = |g, \alpha\rangle = \sum_n C_n |g, n\rangle, \quad C_n = e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}}$$

atom
field
photon number

From Rabi solutions: w/ $\Delta = 0 \Rightarrow$

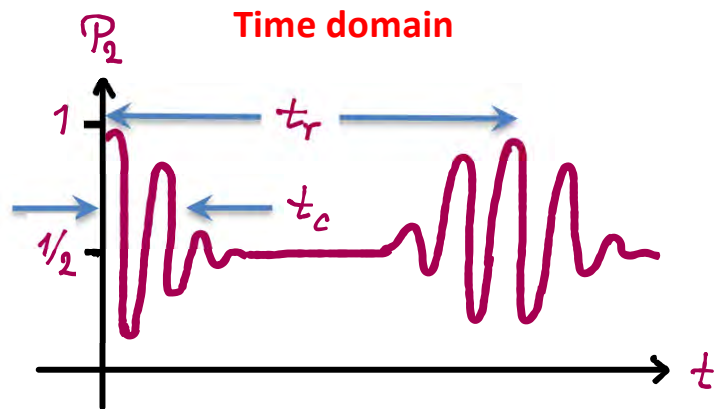
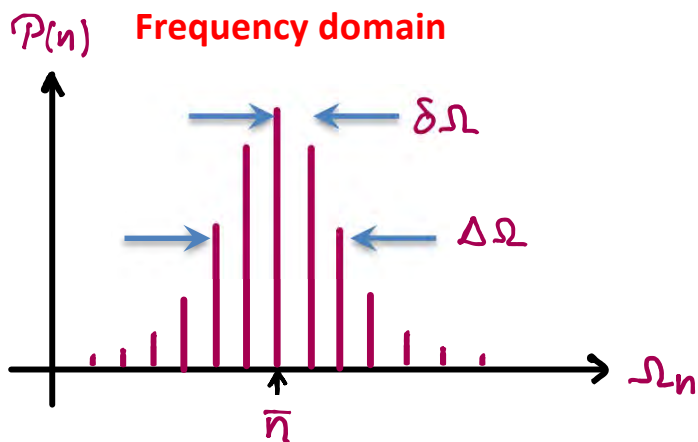
$$C_{g,n} = \cos\left(\frac{\Omega n t}{2}\right), \quad C_{g,n-1} = -i \sin\left(\frac{\Omega n t}{2}\right)$$

Quantized Light – Matter Interactions

- Poisson weighted average of sinusoids
- Sinusoids gradually dephase over time



Collapse of oscillation amplitude



Use $\Delta n = \sqrt{\bar{n}} \Rightarrow \Delta\Omega \sim \Delta\Omega_{\bar{n}+\sqrt{\bar{n}}} - \Delta\Omega_{\bar{n}-\sqrt{\bar{n}}}$

$$t_c = \frac{1}{\Delta\Omega} \sim \frac{1}{2g\sqrt{\bar{n}+\sqrt{\bar{n}}} - 2g\sqrt{\bar{n}-\sqrt{\bar{n}}}} \sim \frac{1}{2g}$$

for $\bar{n} \gg \sqrt{\bar{n}}$

Rephasing: when $(\Omega_{\bar{n}} - \Omega_{\bar{n}-1})t_r \approx 2\pi m$

Similar arguments $\rightarrow$ **Revival time**

$$t_r \sim \frac{2\pi}{\delta\Omega} \sim \frac{2\pi\sqrt{\bar{n}}}{g}$$

Quantized Light – Matter Interactions

Use $\Delta n = \sqrt{\bar{n}} \Rightarrow \Delta \Omega \sim \Delta \Omega_{\bar{n}+\sqrt{\bar{n}}} - \Delta \Omega_{\bar{n}-\sqrt{\bar{n}}}$

$$t_c = \frac{1}{\Delta \Omega} \sim \frac{1}{2g\sqrt{\bar{n}+\sqrt{\bar{n}}} - 2g\sqrt{\bar{n}-\sqrt{\bar{n}}}} \sim \frac{1}{2g}$$

for $\bar{n} \gg \sqrt{\bar{n}}$

Rephasing: when $(\Omega_{\bar{n}} - \Omega_{\bar{n}-1})t_r \approx 2\pi m$

Similar arguments $\rightarrow$ **Revival time**

$$t_r \sim \frac{2\pi}{\delta \Omega} \sim \frac{2\pi\sqrt{\bar{n}}}{g}$$

Collapse & Revival Dynamics

Pure Quantum Phenomenon

(“graininess” of photons)

Classical limit

$$\left\{ \begin{array}{l} g \rightarrow 0 \\ \mathcal{E}_R \rightarrow 0 \\ \frac{\Delta \Omega}{\Omega_{\bar{n}}} \rightarrow 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \bar{n} \rightarrow \infty \\ t_c \rightarrow \infty \\ \Omega_{\bar{n}} \neq 0 \end{array} \right. \text{well defined}$$

$$\Omega_{\bar{n}} = 2g\sqrt{\bar{n}} = \frac{\vec{n}_{21} \cdot 2\vec{\mathcal{E}}_R \mathcal{E}_R \sqrt{\bar{n}}}{\hbar} = \frac{\vec{n}_{21} \cdot \vec{E}}{\hbar}$$

Classical Rabi frequency

mean field $\langle \alpha(t) | \hat{\vec{E}} | \alpha(t) \rangle$

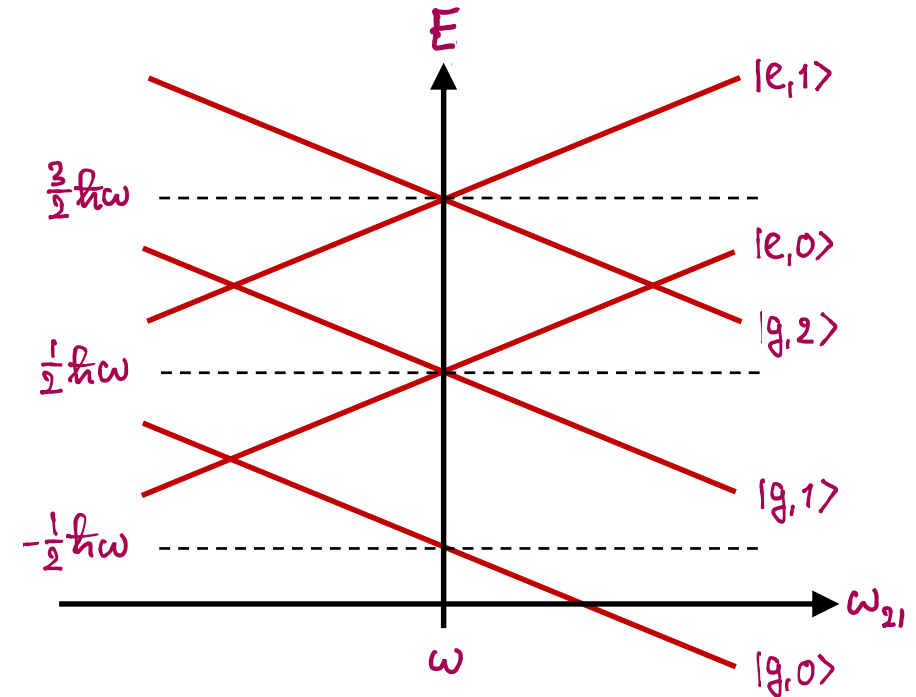
Quantized Light – Matter Interactions

“Bare” states ($g=0$, eigenstates of H_0)

State	Energy
$ g, n\rangle$	$E_{g,n} = -\frac{\hbar\omega_u}{2} + n\hbar\omega$
$ e, n-1\rangle$	$E_{e,n-1} = \frac{\hbar\omega_u}{2} + (n-1)\hbar\omega$

Imagine we can tune $\omega_{2,1}$

Energy level diagram



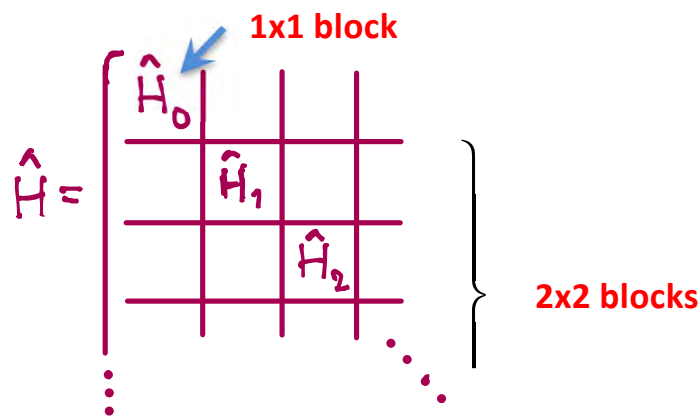
Crossings @ $\omega = \omega_{2,1}$
are degeneracies of
pairs with n shared
excitations

$$\left\{ \begin{array}{ll} n=0 & |g, 0\rangle \\ n=1 & \{|g, 1\rangle, |e, 0\rangle\} \\ n=2 & \{|g, 2\rangle, |e, 1\rangle\} \\ \vdots & \vdots \\ n & \{|g, n\rangle, |e, n-1\rangle\} \end{array} \right.$$

Quantized Light – Matter Interactions

“Dressed” states $\left\{ \begin{array}{l} \text{eigenstates of} \\ \hat{H} = \hat{H}_0 + \hat{H}_{AF} \end{array} \right.$

Structure of $\hat{H}$:



Can write this on the form

$$\hat{H}_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (n - \frac{1}{2}) \hbar \omega + \begin{bmatrix} -\hbar \Delta/2 & \hbar g \sqrt{n} \\ \hbar g \sqrt{n} & \hbar \Delta/2 \end{bmatrix}$$

$$\Delta = \omega_{21} - \omega$$

Eigenvalues $E_{\pm} = (n - \frac{1}{2}) \hbar \omega \pm \frac{\hbar}{2} \sqrt{4g^2 n + \Delta^2}$

Eigenstates

$$|+, n\rangle = \frac{\cos(\Theta_n/2)}{\sin(\Theta_n/2)} |g, n\rangle + \frac{\sin(\Theta_n/2)}{\cos(\Theta_n/2)} |e, n-1\rangle$$

$$|-, n\rangle = -\frac{\sin(\Theta_n/2)}{\cos(\Theta_n/2)} |g, n\rangle + \frac{\cos(\Theta_n/2)}{\sin(\Theta_n/2)} |e, n-1\rangle$$

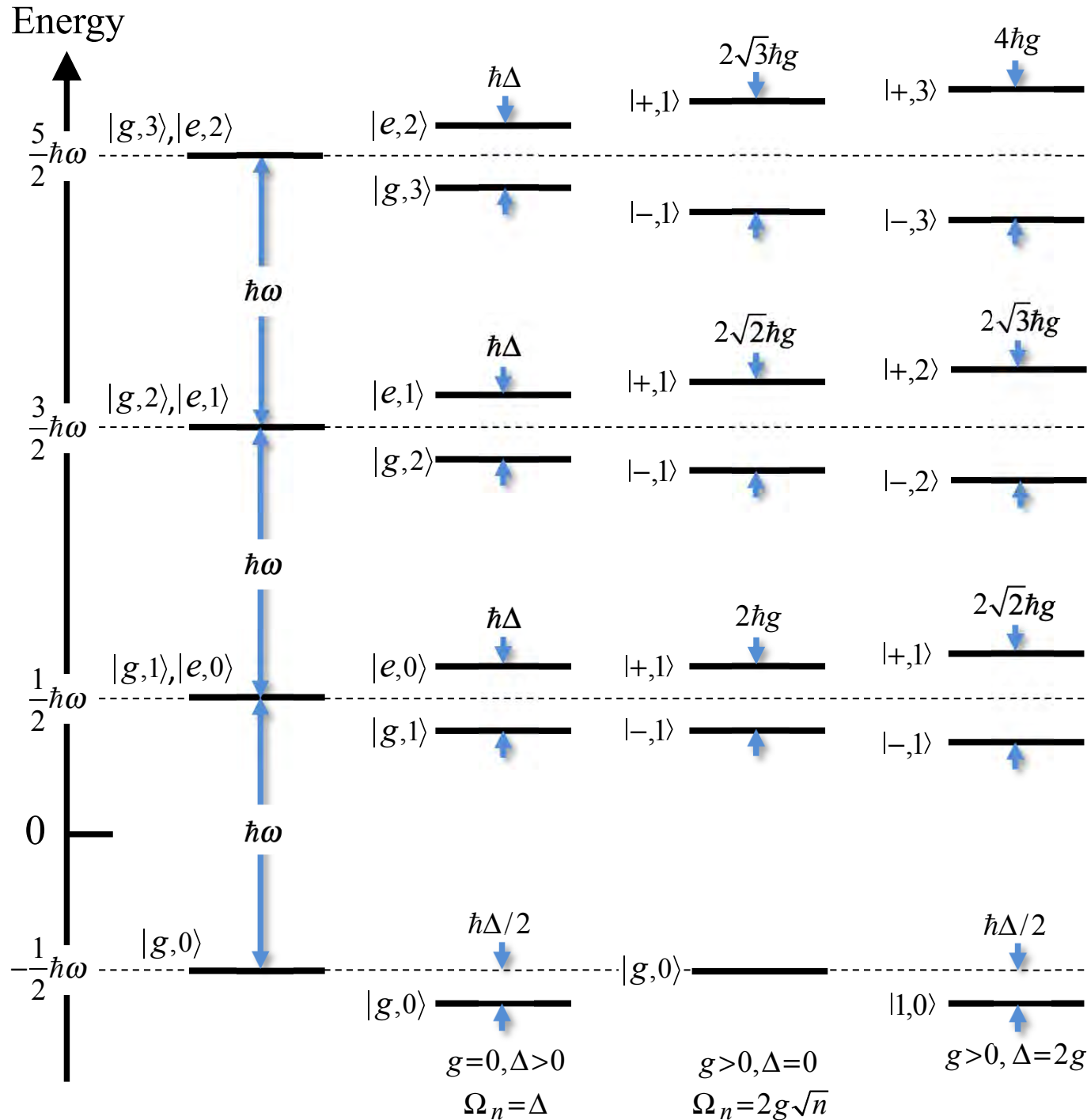
for $\Delta \leq 0$
 $\Delta > 0$

Mixing angle $\tan \Theta_n = -\frac{2g\sqrt{n}}{\Delta}$

Energy Spectrum?

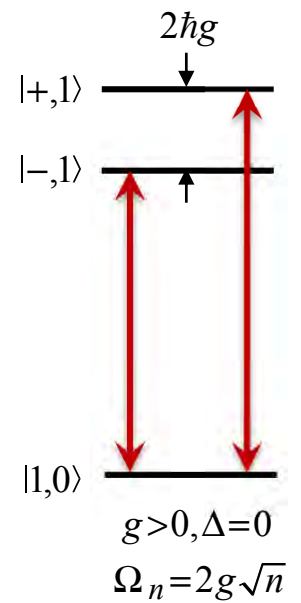
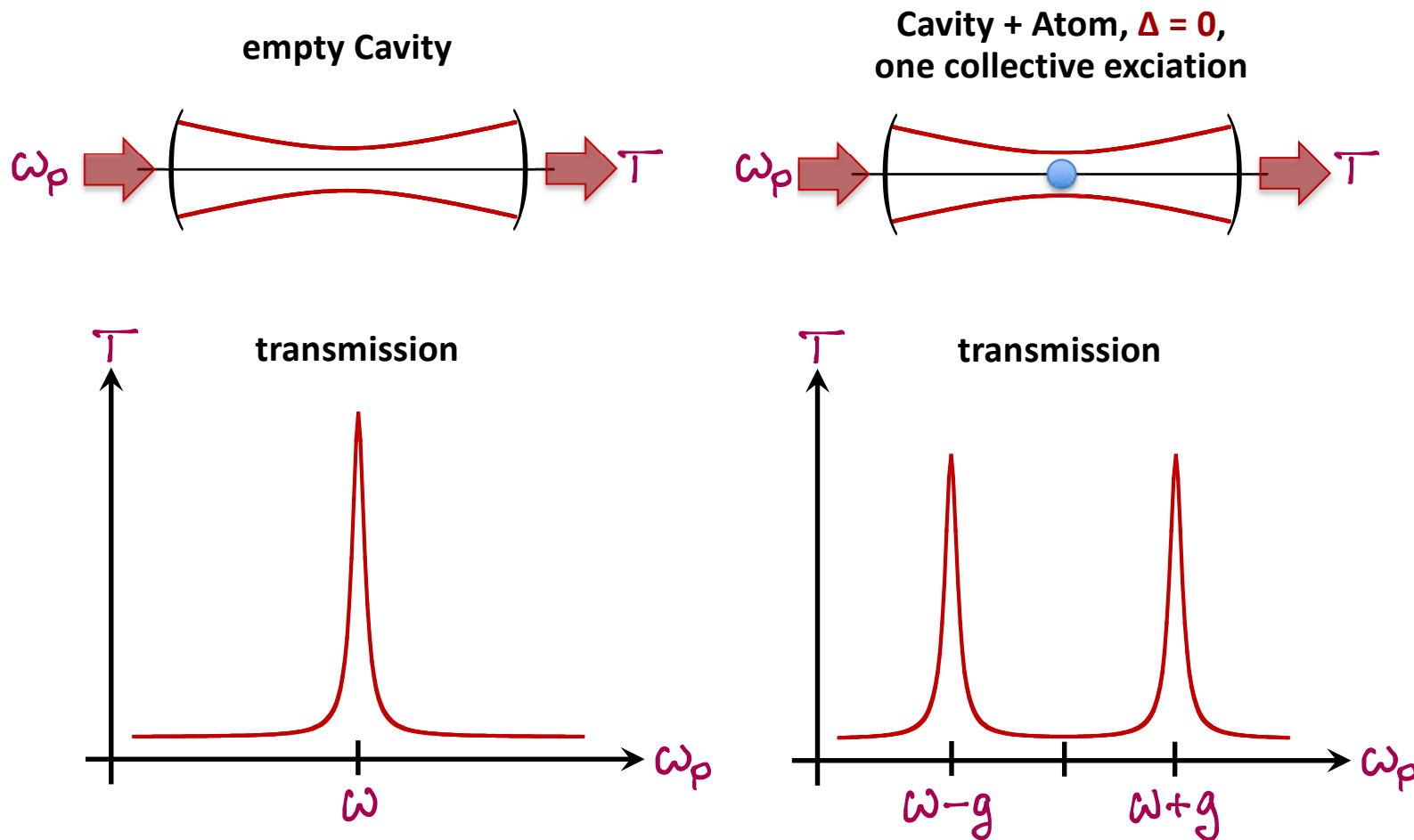
The Jaynes-Cummings Ladder

$$E_{\pm} = \left(n - \frac{1}{2}\right)\hbar\omega \pm \hbar\Omega_n, \quad \Delta = \omega_{eg} - \omega$$



Vacuum Rabi Splitting

Consider the following experiments



Fluorescence - Mollow Triplet

For the coherent state $|\alpha\rangle$ let the mode volume V and mean photon number $\bar{n}$ go to infinity s.t. $V \times \bar{n} \sim \text{constant}$. Then $g \sim \text{constant}$, and thus $\Omega^2 = 4g^2(\bar{n} + \sqrt{\bar{n}}) + \Delta^2 \sim 4g^2\bar{n} + \Delta^2$.

