Quantized Light – Matter Interactions

General Problem:

Quantum Atom $\longleftrightarrow$ Quantized Modes Of EM Field

Electric Dipole Interaction

Starting Point: System Hamiltonian

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} \qquad (1)$$

$$\hat{H}_F = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \left(\hat{a}^+_{\vec{k}}\hat{a}_{\vec{k}} + 1/2\right) \qquad \textbf{Field}$$

$$\hat{H}_A = \sum_i E_i |i\rangle\langle i| = \sum_i E_i \hat{\sigma}_{ii} \qquad \textbf{Atom}$$

$$\hat{H}_{AF} = -\hat{\vec{\mu}} \cdot \hat{\vec{E}}(\vec{r},t) \qquad \textbf{ED interaction}$$

E_i, $|i\rangle$: energies, energy levels of the atom

Dipole Operator:

$$\hat{\vec{\mu}} = \sum_{i,j} \vec{\mu}_{ij} |i\rangle\langle j| = \sum_{i,j} \vec{\mu}_{ij} \hat{\sigma}_{ij} \qquad (2)$$

Field Operator:

$$\hat{\vec{E}}(\vec{r},t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + H.c., \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar\omega_{\vec{k}}}{2\varepsilon_0 V}}$$

2 polarization modes implicit

Pin down atom where $u_{\vec{k}}(\vec{r}) = 1$

– anywhere if $u_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$

– if $u_{\vec{k}}(\vec{r}) = \sin(kz)$ then where $\sin(kz) = 1$

$$\hat{\vec{E}}(\vec{r},t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \left(\hat{a}_{\vec{k}} + \hat{a}^+_{\vec{k}}\right)$$

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Dipole Operator:

(2) $$\boxed{\hat{\vec{\mu}} = \sum_{i,j} \vec{\mu}_{ij} |i\rangle\langle j| = \sum_{i,j} \vec{\mu}_{ij}\, \hat{\sigma}_{ij}}$$

Field Operator:

$$\vec{E}(\vec{r},t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + H.c., \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar\omega_{\vec{k}}}{2\varepsilon_0 V}}$$

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(3) $$\boxed{\hat{\vec{E}}(\vec{r},t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})}$$

Combining (2) & (3):

$$\hat{H}_{AF} = \sum_{ij}\sum_{\vec{k}} -\vec{\mu}_{ij}\cdot\vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}}\, \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})$$

$$= \sum_{ij}\sum_{\vec{k}} \hbar g^{(ij)}_{\vec{k}} \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})$$

where $$g^{(ij)}_{\vec{k}} = \frac{\vec{\mu}_{ij}\cdot\vec{\varepsilon}_{\vec{k}}\mathcal{E}_{\vec{k}}}{\hbar}$$

Rabi Freq., note sign convention

2-level atom ⇨ $(i,j) = (1,2)$:

$$\boxed{\hat{H}_{AF} = \hbar \sum_{\vec{k}} (g_{\vec{k}} \hat{\sigma}_{21} + g^{*}_{\vec{k}} \hat{\sigma}_{12})(\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})}$$

Define:

$$\left.\begin{aligned} \hat{\sigma}_+ &= \hat{\sigma}_{21} = |2\rangle\langle 1| \\ \hat{\sigma}_- &= \hat{\sigma}_{12} = |1\rangle\langle 2| \\ \hat{\sigma}_z &= \hat{\sigma}_{22} - \hat{\sigma}_{11} = |2\rangle\langle 2| - |1\rangle\langle 1| \end{aligned}\right\} \Rightarrow$$

Pauli matrices

$$\hat{\sigma}_x = \tfrac{1}{2}(\hat{\sigma}_+ + \hat{\sigma}_-)$$
$$\hat{\sigma}_y = \tfrac{1}{2i}(\hat{\sigma}_+ - \hat{\sigma}_-)$$
$$\hat{\sigma}_z$$

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With this notation

(4)

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + \cancel{g_{\vec{k}} \hat{\sigma}_+ \hat{a}^+_{\vec{k}}} + \cancel{g_{\vec{k}} \hat{\sigma}_- \hat{a}_{\vec{k}}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right)$$

Energy conservation?

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right)$$

Putting it all together

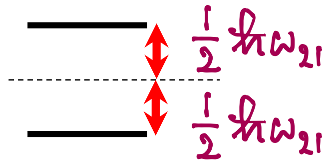
(5)

$$\boxed{\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_z + \sum_{\vec{k}} \hbar \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right)}$$

We changed the zero point for energy by subtracting

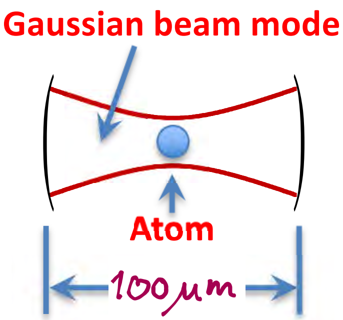
$\sum_{\vec{k}} \frac{1}{2}\hbar\omega_{\vec{k}}$ and $\frac{1}{2}(E_2 - E_1)$

field atom



Interaction with Single-mode Fields

Good approx. in small, high-Q Cavity



$c/2L \gg A_{21}$

$|g_{\vec{k}}| \gg A_{21}, \gamma$

Single-mode (Jaynes-Cummings) Hamiltonian

$$\boxed{\hat{H} = \underbrace{\hbar\omega \hat{a}^+\hat{a} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar g (\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^+)}_{H_{AF}}}$$

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More about Single-mode Cavity QED

$$\hat{H} = \underbrace{\hbar\omega\,\hat{a}^+\hat{a} + \tfrac{1}{2}\hbar\omega_{21}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar(g_{\vec{k}}\hat{\sigma}_+ + g^*_{\vec{k}}\hat{\sigma}_-)(\hat{a}_{\vec{k}} + \hat{a}^+_{\vec{k}})}_{H_{AF}}$$

For simplicity $\vec{\mu}_{21} = \vec{\mu}_{12} \Rightarrow g_{\vec{k}} = g^*_{\vec{k}}$

Note: $\hat{H}_{AF}$ **conserves excitation number, couples** $|2,n\rangle \leftrightarrow |1,n+1\rangle$

Series of 2-level systems, one for each n

All 2-level systems are alike

Rabi problem!

Switch to Interaction Picture:

Sakurai page 318-319

$$\hat{H}_S \to \hat{H}_I = e^{i\frac{\hat{H}_0}{\hbar}t}\hat{H}_{AF}e^{-i\frac{\hat{H}_0}{\hbar}t}$$

$$|\psi_S(t)\rangle \to |\psi_I(t)\rangle = e^{i\frac{\hat{H}_0}{\hbar}t}|\psi_S(t)\rangle$$

$$\hat{H}_{AF} = \hbar g\left(\hat{\sigma}_+\hat{a}e^{i(\omega_{21}-\omega)t} + \hat{\sigma}_+\hat{a}^+e^{i(\omega_{21}+\omega)t} + \hat{\sigma}_-\hat{a}e^{-i(\omega_{21}+\omega)t} + \hat{\sigma}_-\hat{a}^+e^{-i(\omega_{21}-\omega)t}\right)$$

Can show

$$e^{i\omega\hat{a}^+\hat{a}t}\,\hat{a}\,e^{-i\omega\hat{a}^+\hat{a}t} = \hat{a}e^{-i\omega t}$$

$$e^{i\frac{\omega_{21}}{2}\hat{\sigma}_z t}\,\hat{\sigma}_+\,e^{-i\frac{\omega_{21}}{2}\hat{\sigma}_z t} = \hat{\sigma}_+e^{-i\omega_{21}t}$$

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$$\hat{H} = \underbrace{\hbar\omega\,\hat{a}^{+}\hat{a} + \tfrac{1}{2}\hbar\omega_{21}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar(g_{\vec{k}}\hat{\sigma}_+ + g^*_{\vec{k}}\hat{\sigma}_-)(\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})}_{H_{AF}}$$

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RWA and resonant approximation

Jaynes-Cummings Hamiltonian

$$\hat{H}_I = \hat{H}_{AF} = \hbar g\left(\hat{\sigma}_+\hat{a}\,e^{i\Delta t} + \hat{\sigma}_-\hat{a}^{+}e^{-i\Delta t}\right)$$

$\Delta = \omega_{21} - \omega$

Important: Change in notation

For the remainder of this course we change indices 1 to g (ground state) and 2 to e (excited state). Thus a g appearing inside a ket refers to a state, elsewhere g is a Rabi frequency. This is needed for clarity.

Eigenstates of $\hat{H}_0 = \hat{H}_F + \hat{H}_A$

State	Energy
$\|e,n\rangle$	$\hbar\omega n + \frac{1}{2}\hbar\omega_{21}$
$\|g,n+1\rangle$	$\hbar\omega(n+1) - \frac{1}{2}\hbar\omega_{21}$

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$$\hat{H}_{AF} = \hbar g\left(\hat{\sigma}_+\hat{a}e^{i(\omega_{21}-\omega)t} + \hat{\sigma}_+\hat{a}^+e^{i(\omega_{21}+\omega)t} + \hat{\sigma}_-\hat{a}e^{-i(\omega_{21}+\omega)t} + \hat{\sigma}_-\hat{a}^+e^{-i(\omega_{21}-\omega)t}\right)$$

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Cavity QED version of the Rabi Problem

$$\lvert\psi(0)\rangle = \lvert e,n\rangle$$

$$\lvert\psi(t)\rangle = C_{g,n+1}\lvert g,n+1\rangle + C_{e,n}\lvert e,n\rangle$$

Matrix elements

$$\langle e,n\rvert\hat{H}_{AF}\lvert g,n+1\rangle = \hbar g\sqrt{n+1}\,e^{i\Delta t}$$

$$\langle g,n+1\rvert\hat{H}_{AF}\lvert e,n\rangle = \hbar g\sqrt{n+1}\,e^{-i\Delta t}$$

Schrödinger Equation

$$i\hbar\frac{d}{dt}\begin{pmatrix} C_{g,n+1} \\ C_{e,n}\end{pmatrix} = \hbar g\sqrt{n+1}\begin{pmatrix} 0 & e^{-i\Delta t} \\ e^{i\Delta t} & 0\end{pmatrix}\begin{pmatrix} C_{g,n+1} \\ C_{e,n}\end{pmatrix}$$

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$$\dot{C}_{g,n+1} = -ig\sqrt{n+1}\,e^{-i\Delta t}\,C_{e,n}$$

$$\dot{C}_{e,n} = -ig\sqrt{n+1}\,e^{i\Delta t}\,C_{g,n+1}$$

Substitute $C_{g,n+1} \to C_e$, $C_{e,n} \to C_e e^{i\Delta t}$

Looks exactly like Semiclassical Rabi problem

Solve for $C_g(0) = 0$, $C_e(0) = 1$

$$C_{e,n}(t) = \left[\cos\left(\frac{\Omega_n t}{2}\right) - i\frac{\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]e^{i\Delta t/2}$$

$$C_{g,n+1} = -i\,\frac{2g\sqrt{n+1}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)e^{-i\Delta t/2}$$

$$\Omega_n = \left(4g^2(n+1) + \Delta^2\right)^{1/2}$$

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$$\dot{c}_{g,n+1} = -ig\sqrt{n+1}\, e^{-i\Delta t} c_{e,n}$$

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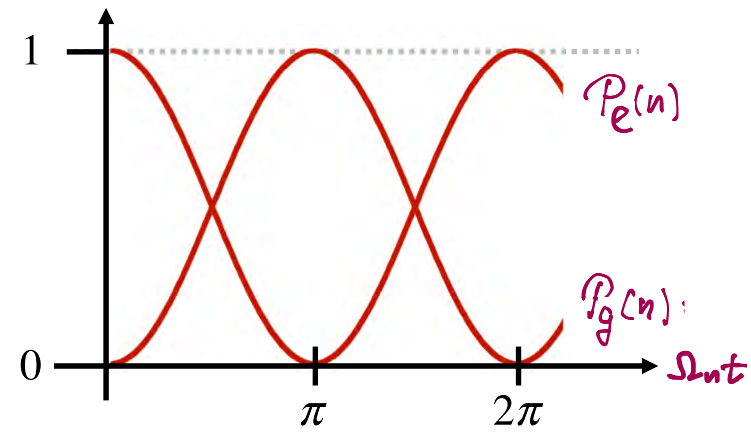
Rabi Oscillations

$$P_e(n) = \cos^2\left(\frac{\Omega_n t}{2}\right) + \left(\frac{\Delta}{\Omega_n}\right)^2 \sin^2\left(\frac{\Omega_n t}{2}\right)$$

$$P_g(n) = \frac{4g^2(n+1)}{\Omega_n^2}\sin^2\left(\frac{\Omega_n t}{2}\right)$$

Example: $\Delta = 0$



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Vacuum Rabi Oscillations

If $|\psi(0)\rangle = |e,0\rangle$ ⇨ no photons in field

yet $|e,0\rangle$ evolves into $|g,1\rangle$

Uniquely QED phenomenon!

Asymmetry $\begin{cases} |e,n=0\rangle \longrightarrow |g,n=1\rangle \\ |g,n=0\rangle \longrightarrow |g,n=1\rangle \end{cases}$

holds germ **of Spontaneous Decay!**

Quantized Light – Matter Interactions

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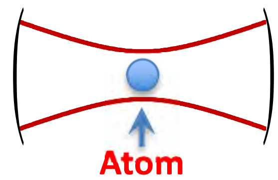
Asymmetry
$$\begin{cases} |e,n=0\rangle \longrightarrow |g,n=1\rangle \\ |g,n=0\rangle \not\longrightarrow |g,n=1\rangle \end{cases}$$

holds germ **of Spontaneous Decay!**

Next: More Cavity QED



2-level atom

Single cavity mode

What happens with a Coherent State in the Cavity mode?

(Quantum-Classical correspondence)

Initial atom-field state:

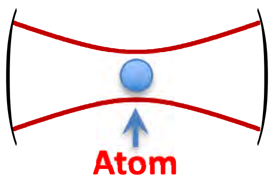
$$|\psi(0)\rangle = \underbrace{|g\rangle}_{\text{atom}} \otimes \underbrace{|\alpha\rangle}_{\text{field}} = \sum_n C_n |g,n\rangle, \quad C_n = e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}}$$

From Rabi solutions: $\Delta = 0$ ⇨

$$C_{g,n} = \cos\left(\frac{\Omega_n t}{2}\right), \quad C_{g,n-1} = -i\sin\left(\frac{\Omega_n t}{2}\right)$$

Quantized Light – Matter Interactions

Today: More Cavity QED



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atom field

From Rabi solutions: $\Delta = 0$ ➡

$$C_{g,n} = \cos\left(\frac{\Omega_n t}{2}\right) \quad ➡ \quad C_{e,n-1} = -i \sin\left(\frac{\Omega_n t}{2}\right)$$

Therefore

uncoupled

$$\boxed{|\psi(t)\rangle = C_0 |g,0\rangle + \sum_{n=1}^{\infty} C_n \left[\cos\left(\frac{\Omega_n t}{2}\right)|g,n\rangle - i \sin\left(\frac{\Omega_n t}{2}\right)|e,n-1\rangle\right]}$$

Consider the Atomic Excited State Population

$$P_e(t) = \sum_{n=0}^{\infty} P_{e,n} = \sum_{n=0}^{\infty} |\langle e,n|\psi(t)\rangle|^2$$

$$= \sum_{n=0}^{\infty} |C_n|^2 \sin^2\left(\frac{\Omega_n t}{2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} \sin^2\left(\frac{\Omega_n t}{2}\right)$$

Use $|\alpha|^2 = \bar{n}$ and $\Omega_n = 2g\sqrt{n}$

$$\boxed{P_e(t) = \sum_{n=0}^{\infty} \frac{(\bar{n})^n e^{-\bar{n}}}{n!} \sin^2(g\sqrt{n}\, t)}$$

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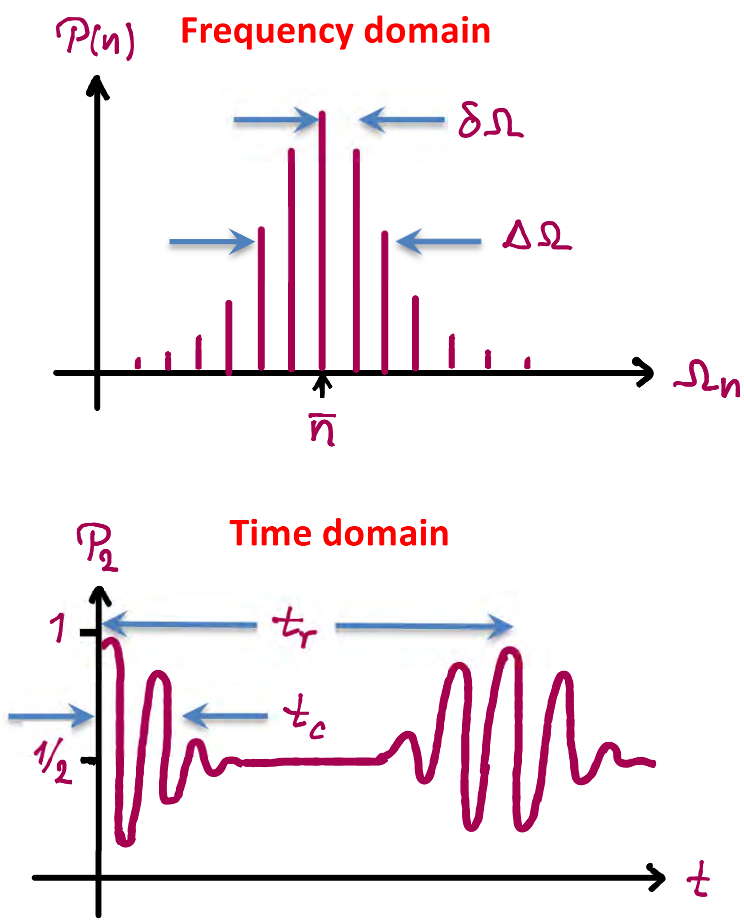
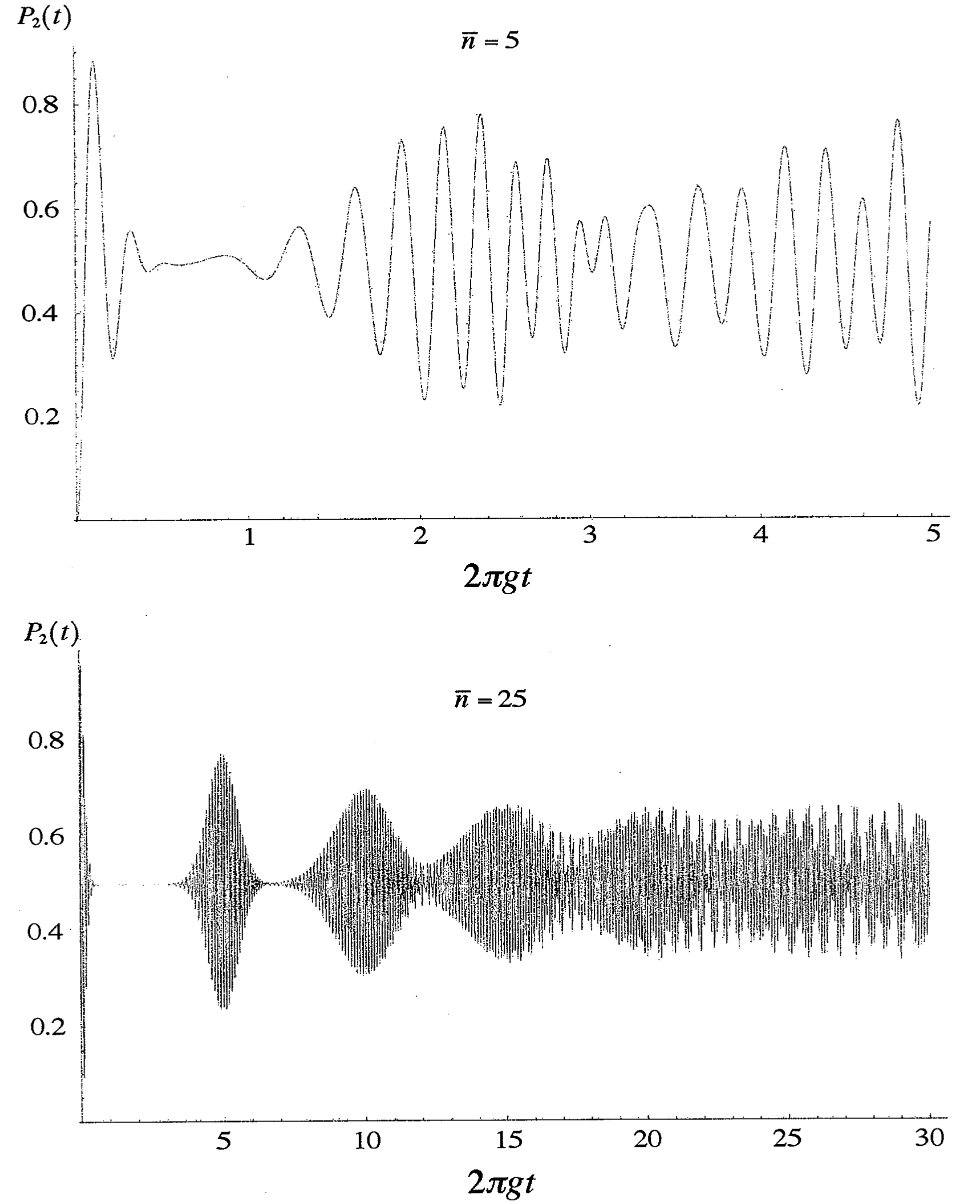
– Poisson weighted average of sinusoids
– Sinusoids gradually dephase over time

Collapse of oscillation amplitude



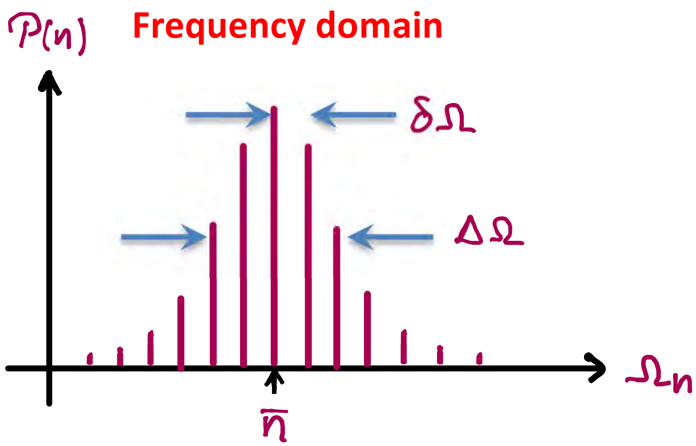
Numerical examples

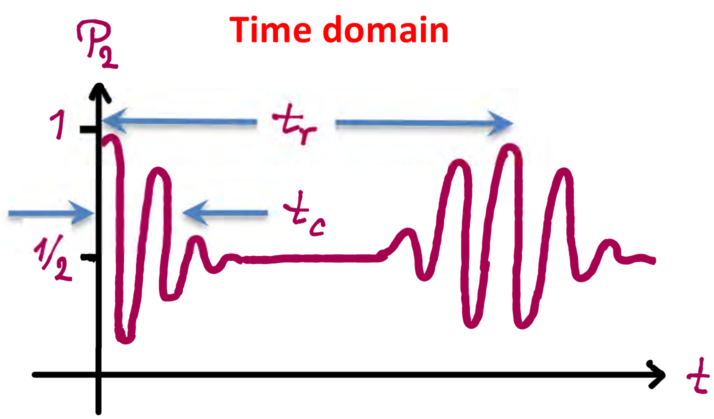


Quantized Light – Matter Interactions

– **Poisson weighted average of sinusoids**
– **Sinusoids gradually dephase over time**

Collapse of oscillation amplitude





Use $\Delta n = \sqrt{\bar n}$ $\Rightarrow$ $\Delta\Omega \sim \Delta\Omega_{\bar n+\sqrt{\bar n}} - \Delta\Omega_{\bar n-\sqrt{\bar n}}$

$$\boxed{t_c = \frac{1}{\Delta\Omega} \sim \frac{1}{2g\sqrt{\bar n+\sqrt{\bar n}} - 2g\sqrt{\bar n-\sqrt{\bar n}}} \sim \frac{1}{2g}}$$

for $\bar n \gg \sqrt{\bar n}$

Rephasing: **when** $(\Omega_{\bar n} - \Omega_{\bar n-1})\, t_r \simeq 2\pi m$

Similar arguments -> Revival time

$$\boxed{t_r \sim \frac{2\pi}{\delta\Omega} \sim \frac{2\pi\sqrt{\bar n}}{g}}$$

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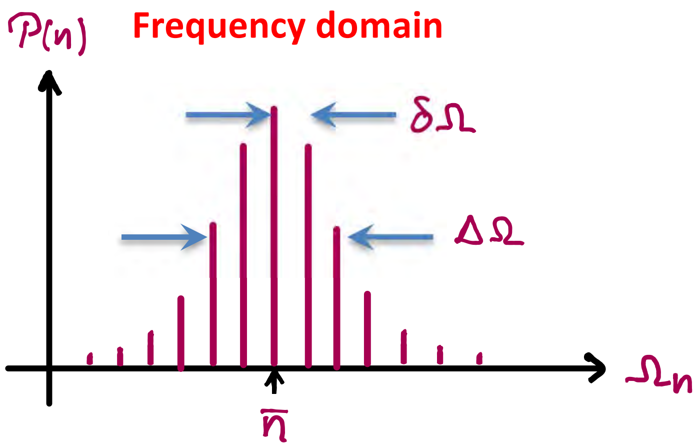

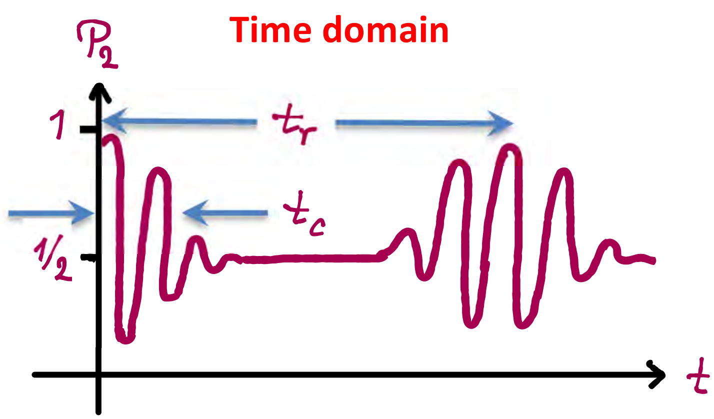

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Quantized Light – Matter Interactions

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Rephasing: **when** $(\Omega_{\bar{n}} - \Omega_{\bar{n}-1})t_r \simeq 2\pi m$

Similar arguments -> Revival time

$$\boxed{t_r \sim \frac{2\pi}{\delta\Omega} \sim \frac{2\pi\sqrt{\bar{n}}}{g}}$$

Collapse & Revival Dynamics

Pure Quantum Phenomenon

("graininess" of photons)

Classical limit $\left\{\begin{array}{l} g \to 0 \\ \mathcal{E}_k \to 0 \\ \dfrac{\Delta\Omega}{\Omega_{\bar{n}}} \to 0 \end{array}\right.$ $\Rightarrow$ $\bar{n} \to \infty$, $t_c \to \infty$; $\Rightarrow$ $\Omega_{\bar{n}} \neq 0$ **well defined**

$$\boxed{\Omega_{\bar{n}} = 2g\sqrt{\bar{n}} = \frac{\vec{\mu}_{21}\cdot 2\vec{\mathcal{E}}_k \mathcal{E}_k \sqrt{\bar{n}}}{\hbar} = \frac{\vec{\mu}_{21}\cdot\vec{E}}{\hbar}}$$

Classical Rabi frequency

mean field $\langle\alpha(t)|\hat{\vec{E}}|\alpha(t)\rangle$

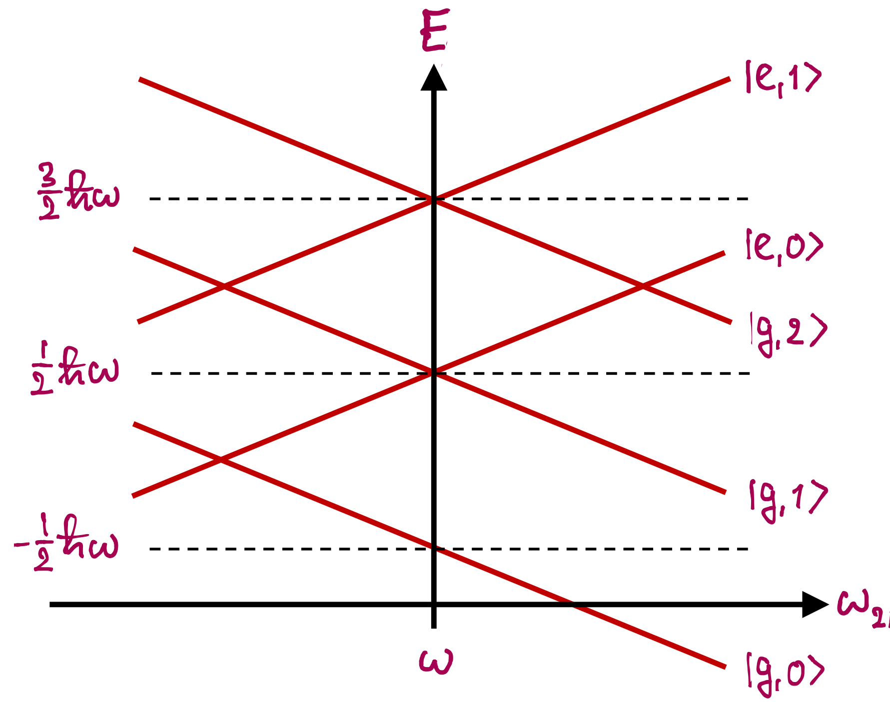
Quantized Light – Matter Interactions

"Bare" states ($g=0$, eigenstates of H_0)

State	Energy
$\|g,n\rangle$	$E_{g,n} = -\frac{\hbar\omega_{21}}{2} + n\hbar\omega$
$\|e,n-1\rangle$	$E_{e,n-1} = \frac{\hbar\omega_{21}}{2} + (n-1)\hbar\omega$

Imagine we can tune ω_{21}

Energy level diagram



Crossings @ $\omega = \omega_{21}$ are degeneracies of pairs with n shared excitations

$n=0$ $\quad |g,0\rangle$

$n=1$ $\quad \{|g,1\rangle, |e,0\rangle\}$

$n=2$ $\quad \{|g,2\rangle, |e,1\rangle\}$

$\vdots$ $\quad \vdots$

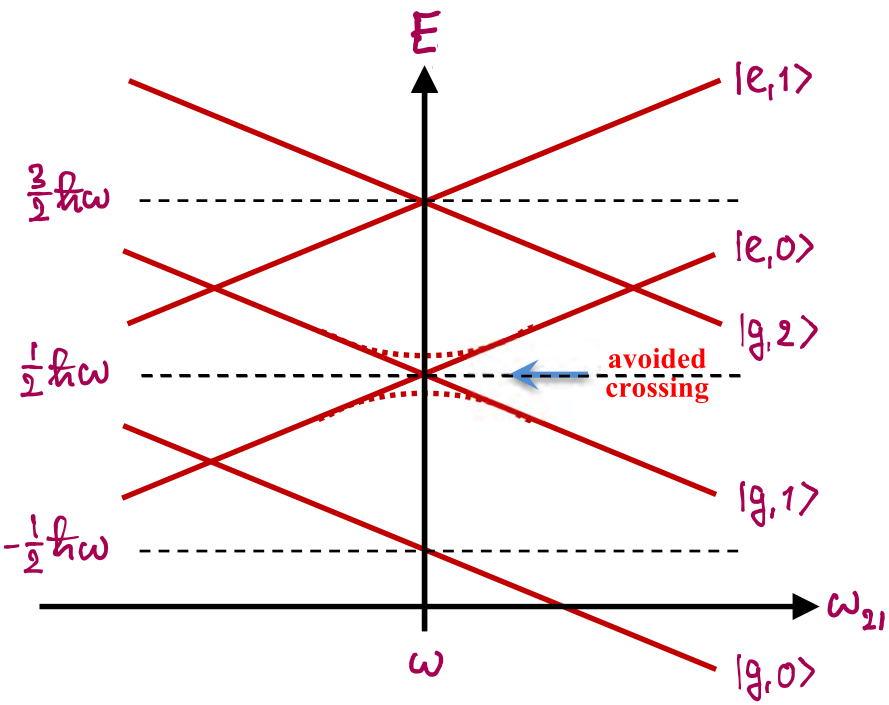
n $\quad \{|g,n\rangle, |e,n-1\rangle\}$

Quantized Light – Matter Interactions

"Bare" states ($g=0$, eigenstates of H_0)

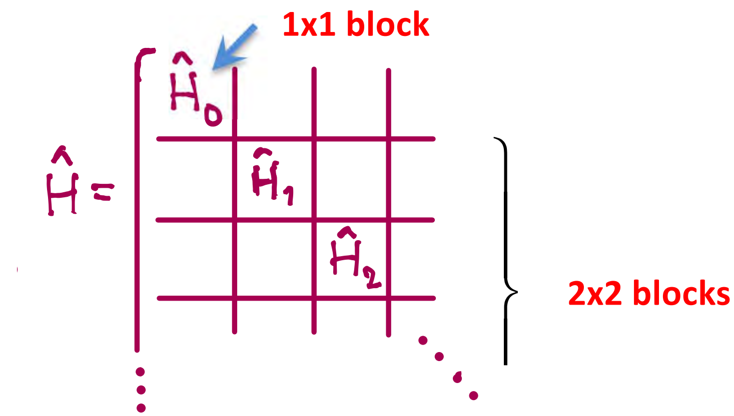
State	Energy
$\|g,n\rangle$	$E_{g,n} = -\frac{\hbar\omega_{21}}{2} + n\hbar\omega$
$\|e,n-1\rangle$	$E_{e,n-1} = \frac{\hbar\omega_{21}}{2} + (n-1)\hbar\omega$

Energy level diagram



"Dressed" states $\left\{ \begin{array}{l} \text{eigenstates of} \\ \hat{H} = \hat{H}_0 + \hat{H}_{AF} \end{array} \right.$

Structure of $\hat{H}$:



Can write this on the form

$$\boxed{\hat{H}_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (n-\tfrac{1}{2})\hbar\omega + \begin{bmatrix} -\hbar\Delta/2 & \hbar g\sqrt{n} \\ \hbar g\sqrt{n} & \hbar\Delta/2 \end{bmatrix}}$$

$\Delta = \omega_{21} - \omega$

Quantized Light – Matter Interactions

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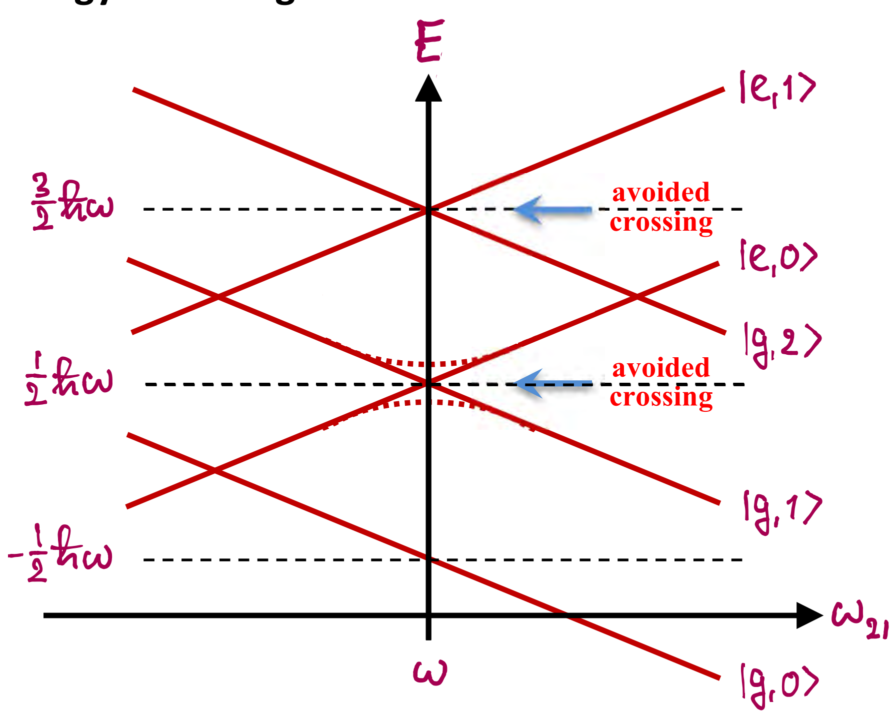
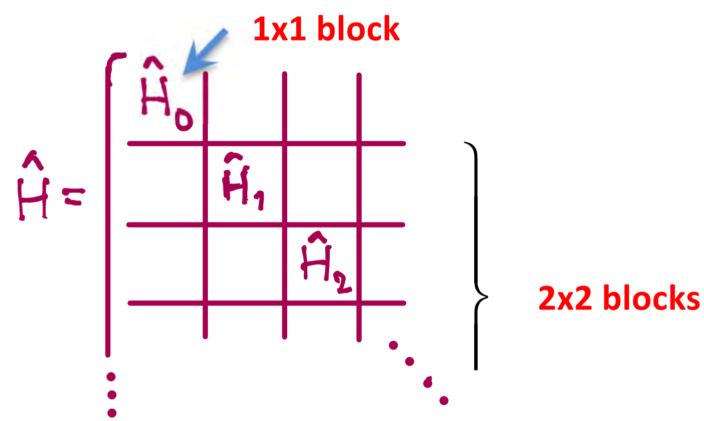
Energy level diagram



"Dressed" states — eigenstates of $\hat{H} = \hat{H}_0 + \hat{H}_{AF}$

Structure of $\hat{H}$:



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Quantized Light – Matter Interactions

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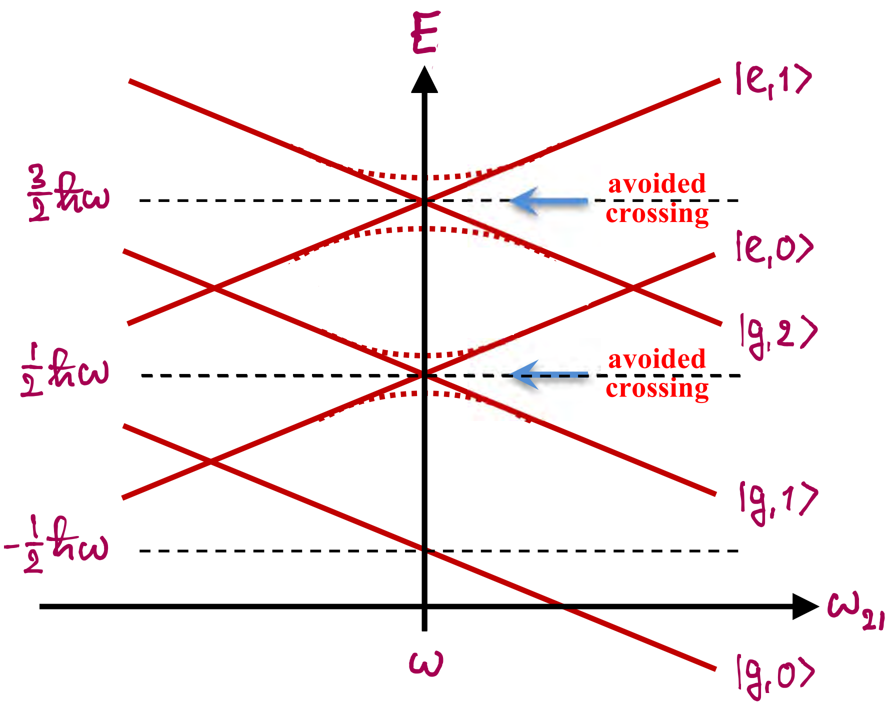
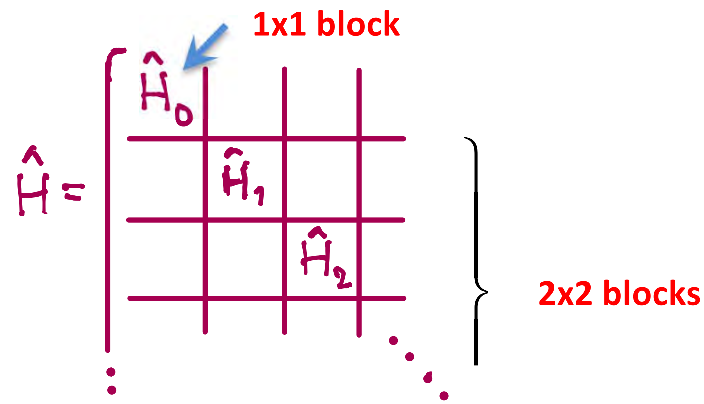
Energy level diagram



"Dressed" states: eigenstates of $\hat{H} = \hat{H}_0 + \hat{H}_{AF}$

Structure of $\hat{H}$:



Can write this on the form

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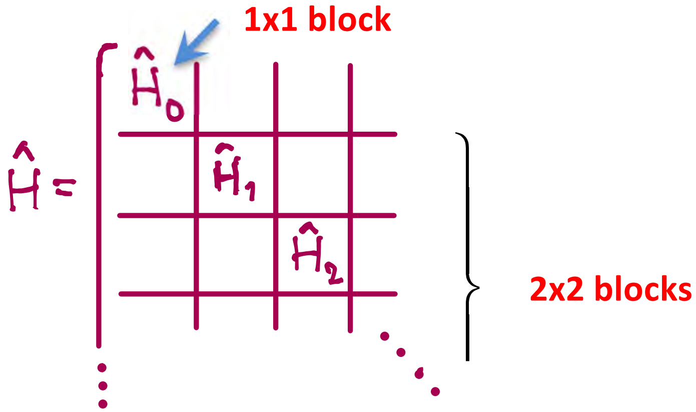
Quantized Light – Matter Interactions

"Dressed" states $\left\{ \begin{array}{l} \text{eigenstates of} \\ \hat{H} = \hat{H}_0 + \hat{H}_{AF} \end{array} \right.$

Structure of $\hat{H}$:



Can write this on the form

$$\boxed{\hat{H}_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (n-\tfrac{1}{2})\hbar\omega + \begin{bmatrix} -\hbar\Delta/2 & \hbar g\sqrt{n} \\ \hbar g\sqrt{n} & \hbar\Delta/2 \end{bmatrix}}$$

$\Delta = \omega_{21} - \omega$

Eigenvalues $E_\pm = (n-\frac{1}{2})\hbar\omega \pm \frac{\hbar}{2}\sqrt{4g^2 n + \Delta^2}$

Eigenstates

$$|+,n\rangle = \begin{matrix}\cos \\ \sin\end{matrix}\left(\theta_n/2\right)|g,n\rangle + \begin{matrix}\sin \\ \cos\end{matrix}\left(\theta_n/2\right)|e,n-1\rangle$$

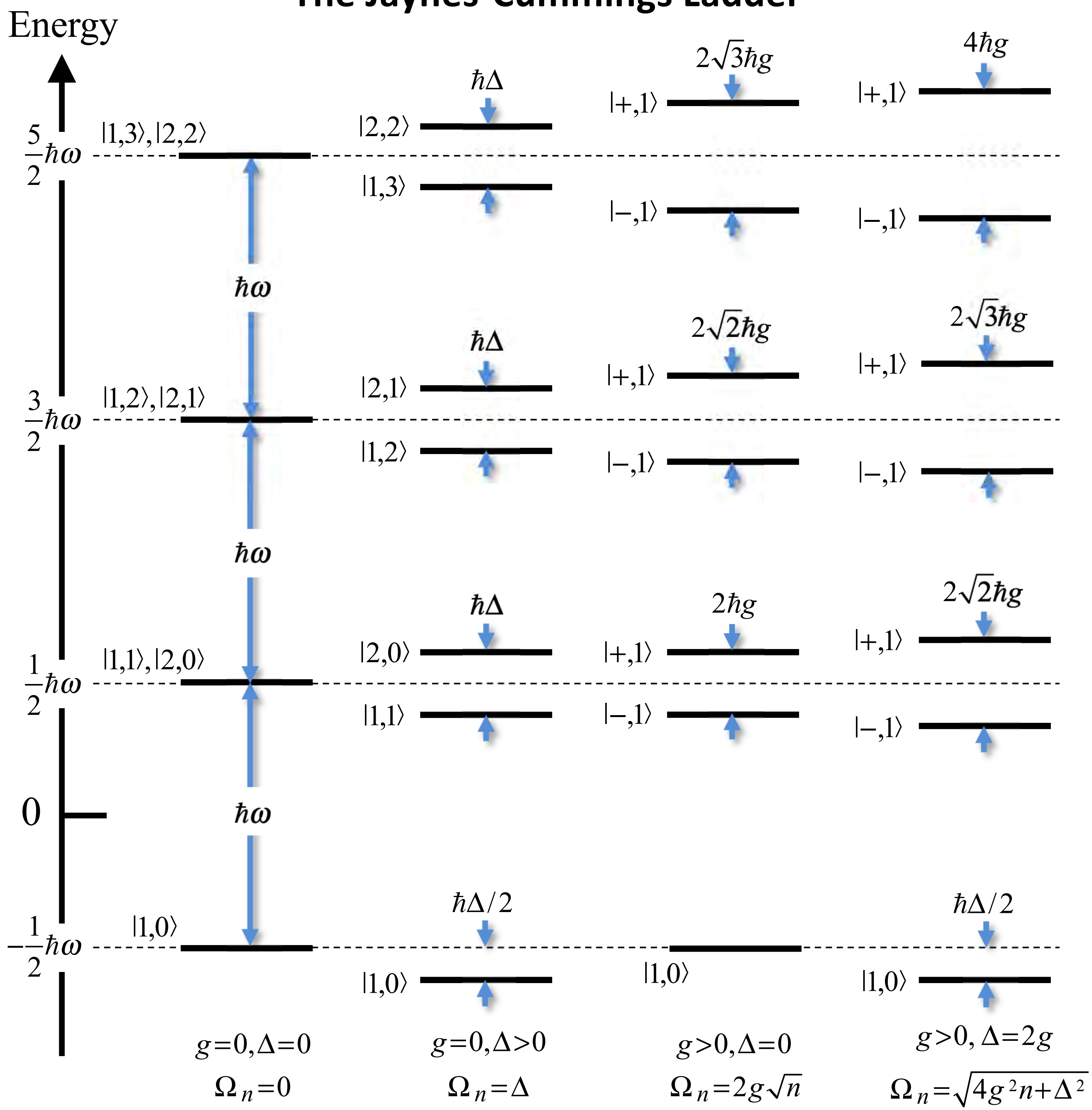
$$|-,n\rangle = -\begin{matrix}\sin \\ \cos\end{matrix}\left(\theta_n/2\right)|g,n\rangle + \begin{matrix}\cos \\ \sin\end{matrix}\left(\theta_n/2\right)|e,n-1\rangle$$

for $\begin{matrix}\Delta \leq 0 \\ \Delta > 0\end{matrix}$

Mixing angle $\tan\theta_n = -\dfrac{2g\sqrt{n}}{\Delta}$

Energy Spectrum?

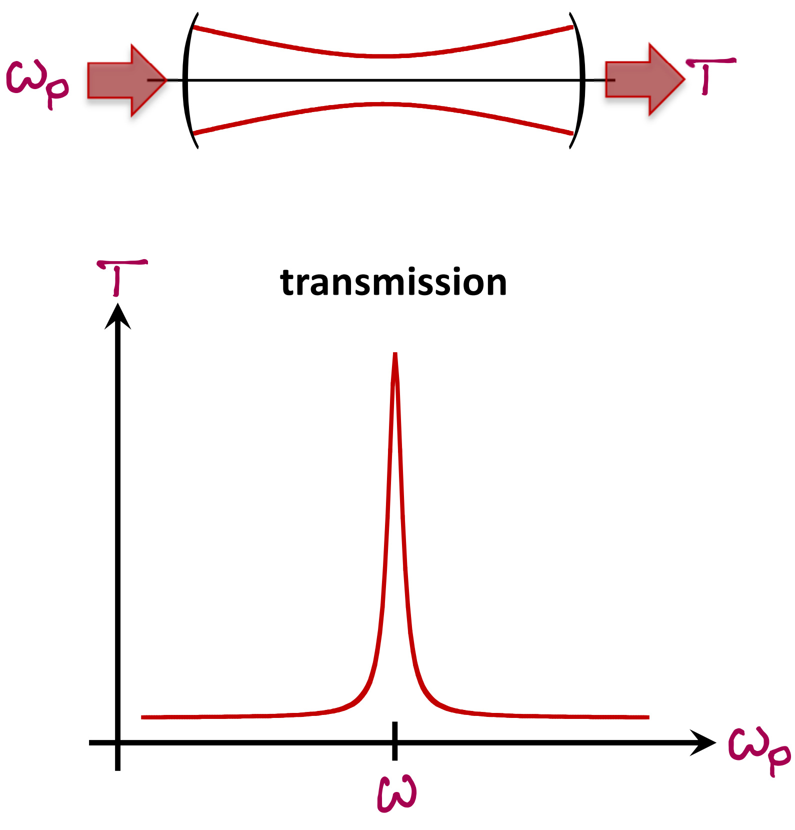
The Jaynes-Cummings Ladder



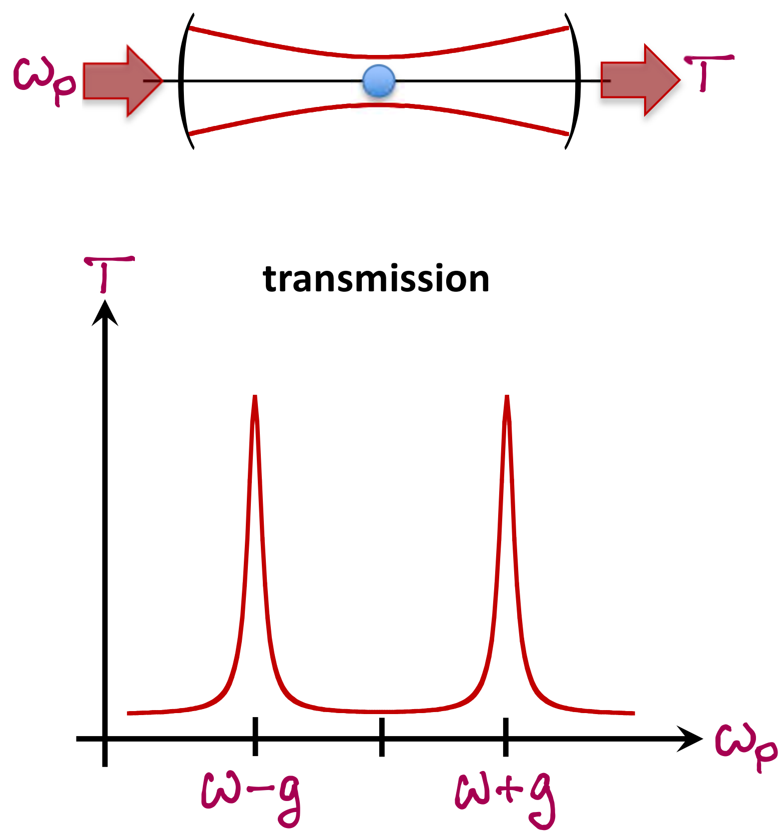
Vacuum Rabi Splitting

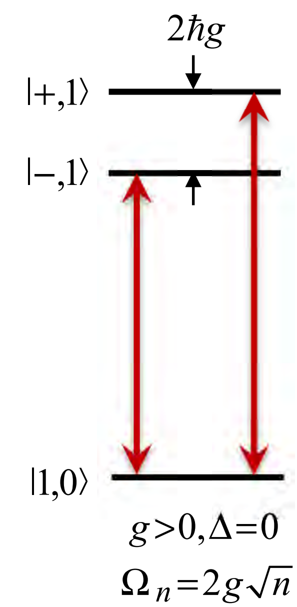
Consider the following experiments

empty Cavity



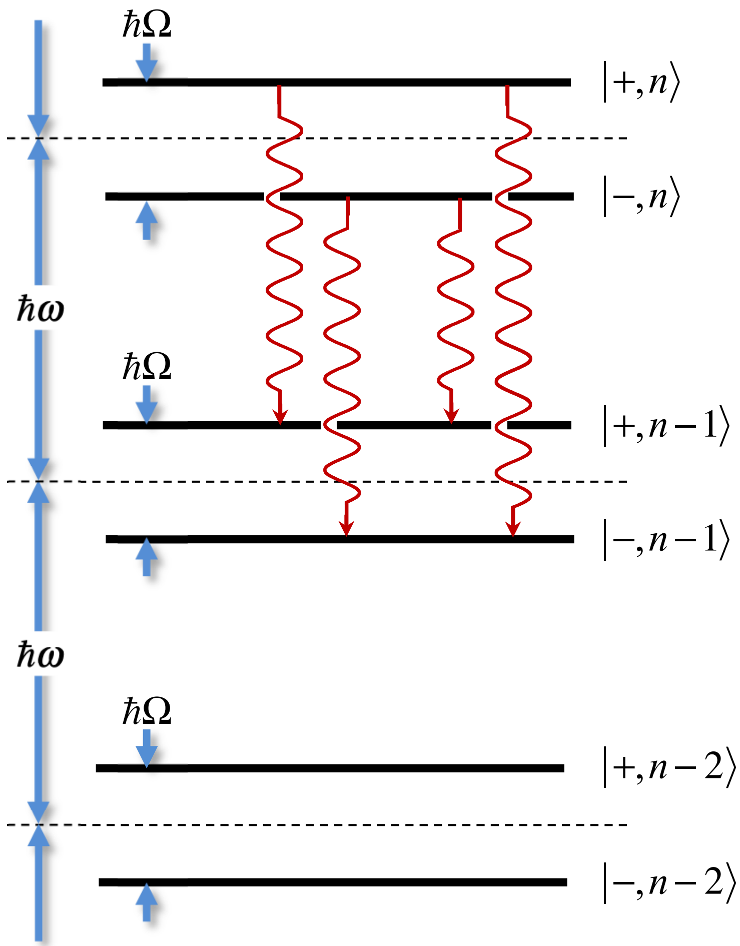
Cavity + Atom, Δ = 0,
one collective exciation





Fluorescence - Mollow Triplet

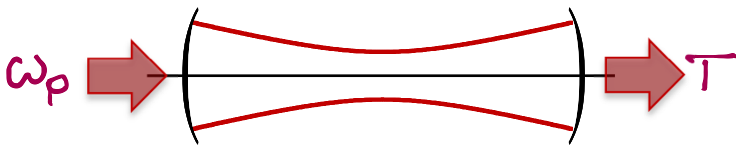
For the coherent state $|\alpha\rangle$ **let the mode volume** V **and mean photon number** $\bar{n}$ **go to infinity s.t.** $V \times \bar{n} \sim$ **constant. Then** $g \sim$ **constant, and thus** $\Omega^2 = 4g^2(\bar{n} + \sqrt{\bar{n}}) + \Delta^2 \sim 4g^2\bar{n} + \Delta^2$ **.**



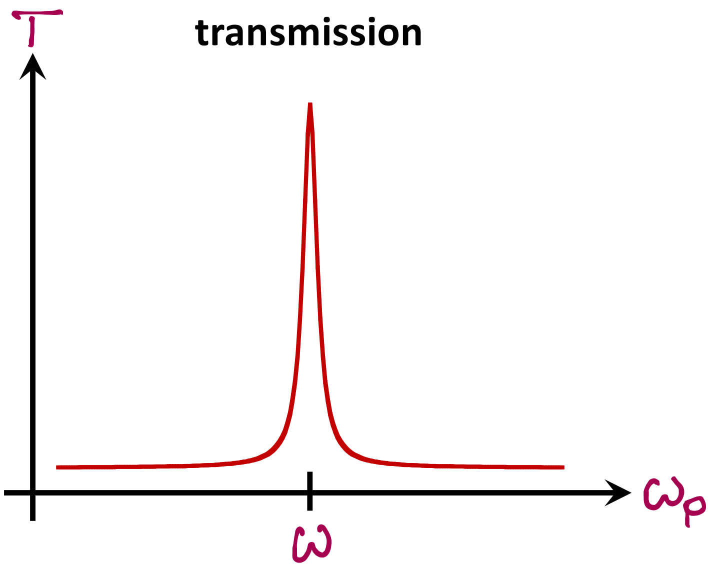
Vacuum Rabi Splitting

Consider the following experiments

empty Cavity



transmission



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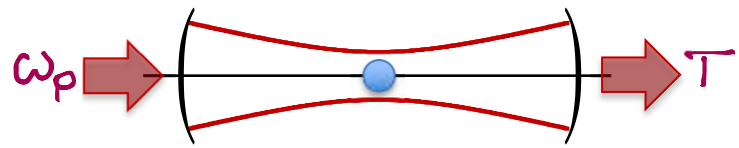

transmission