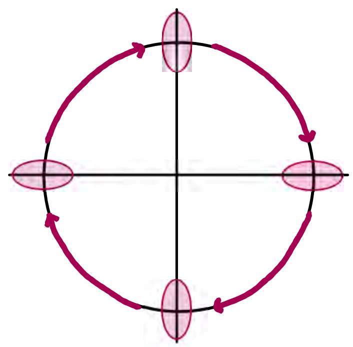
Quantum States of the Quantized Field

Squeezed States

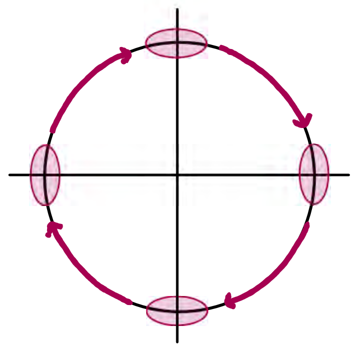
Minimum uncertainty states w/assymmetry

$$\Delta X \, \Delta Y = 1/4 \, , \quad \Delta X(t) \neq \Delta Y(t)$$

Phase Squeezing



Amplitude Squeezing



Requires interaction with Nonlinear medium

Odds and Ends – Thermal States

$$\hat{\rho} = \sum_n P(n) |n\rangle\langle n| = \frac{1}{Z} \sum_n e^{-E_n/k_B T} |n\rangle\langle n|$$

Quantum States of the Quantized Field

Odds and Ends – Thermal States

$$Z = \mathrm{Tr}\left[e^{-\hat{H}/k_B T}\right]$$

$$\hat{\rho} = \sum_n P(n)|n\rangle\langle n| = \frac{1}{Z}\sum_n e^{-E_n/k_B T}|n\rangle\langle n|$$

$$= (1-q)\sum_n q^n |n\rangle\langle n|, \quad q = e^{-\hbar\omega/k_B T}$$

Mean Photon Number:

$$\bar{n} = \mathrm{Tr}(\hat{\rho}\hat{N}) = \sum_{k,n}\langle k|(1-q)q^n|n\rangle\langle n|\hat{N}|k\rangle$$

$$= (1-q)\sum_n n q^n = \frac{q}{1-q}$$

Photon Number Uncertainty:

$$\langle \hat{N}^2\rangle = (1-q)\sum_n n^2 q^n = \frac{q^2+q}{(1-q)}$$

$$\Delta n^2 = \langle \hat{N}^2\rangle - \langle \hat{N}\rangle^2$$

$$= \frac{q^2+q}{(1-q)} - \frac{q^2}{(1-q)^2} = \frac{q}{(1-q)^2}$$

$$\bar{n} = \frac{q}{1-q}$$

$$\Delta n = \frac{\sqrt{q}}{1-q} = \sqrt{\bar{n}(\bar{n}+1)} \geq \sqrt{\bar{n}}$$

Coherent State limit

Optical Frequencies, Room Temperature:

$$\lambda = 1\,\mu m, \qquad T = 300\,K$$

$$q = 6.5\times 10^{-6}, \quad \bar{n} \sim 10^{-6}$$

Quantum States of the Quantized Field

Odds and Ends – Quantum-Classical Correspondence

Define a Translation Operator

$$\hat{T}_\alpha(t) = e^{\alpha^* e^{i\omega t}\hat{a} - \alpha e^{-i\omega t}\hat{a}^+} = \hat{D}(-\alpha e^{-i\omega t})$$

Use $[\hat{a}, \hat{F}(\hat{a}^+)] = dF(\hat{a}^+)/d\hat{a}^+$ **to show**

$$[\hat{a}, \hat{T}_\alpha] = \hat{a}\hat{T}_\alpha - \hat{T}_\alpha\hat{a} = -\alpha e^{-i\omega t}\hat{T}_\alpha$$

$$\Rightarrow \hat{T}_\alpha\hat{a} = \hat{a}\hat{T}_\alpha + \alpha e^{-i\omega t}\hat{T}_\alpha$$

$$\Rightarrow \hat{T}_\alpha\hat{a}\hat{T}_\alpha^+ = \hat{a} + \alpha e^{-i\omega t}$$

From this we get

(1) Field Observable

$$\hat{E}'_\perp = \underbrace{\hat{T}_\alpha \hat{E}_\perp \hat{T}_\alpha^+}_{} = \hat{T}_\alpha\left(\mathcal{E}_k \hat{a} e^{i\vec{k}\cdot\vec{r}} + H.C.\right)\hat{T}_\alpha^+$$

$$= \mathcal{E}_k \hat{a} e^{i\vec{k}\cdot\vec{r}} + H.C. + \mathcal{E}_k \alpha e^{-i(\omega t - \vec{k}\cdot\vec{r})} + C.C.$$

$$= \hat{E}_\perp + E_\perp^{Cl}(\alpha, t)$$

(2) Classical Field

(3) Field Observable

We also have $|\psi'(t)\rangle = \hat{T}_\alpha |\alpha(t)\rangle = |0\rangle$

Action of the unitary transformation $\hat{T}_\alpha(t)$

$$\boxed{\begin{aligned} &\hat{E}'_\perp = \hat{T}_\alpha(t)\hat{E}_\perp \hat{T}_\alpha(t)^+ = \hat{E}_\perp + E_\perp^{Cl}(\alpha, t) \\ &|\psi'(t)\rangle = T_\alpha(t)|\alpha(t)\rangle = |0\rangle \end{aligned}}$$

We can work with

$\hat{E}_\perp, |\alpha(t)\rangle$ **or** $\hat{E}_\perp + E_\perp^{Cl}(\alpha, t), |0\rangle$

Validates Semiclassical Optics for strong Coherent Fields!

Quantized Light – Matter Interactions

General Problem:

Quantum Atom ⟷ Quantized Modes Of EM Field

Electric Dipole Interaction

Starting Point: System Hamiltonian

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} \qquad (1)$$

$$\hat{H}_F = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \left(\hat{a}^+_{\vec{k}}\hat{a}_{\vec{k}} + \tfrac{1}{2}\right) \qquad \text{Field}$$

$$\hat{H}_A = \sum_i E_i |i\rangle\langle i| = \sum_i E_i \hat{\sigma}_{ii} \qquad \text{Atom}$$

$$\hat{H}_{AF} = -\hat{\vec{\mu}}\cdot\hat{\vec{E}}(\vec{r},t) \qquad \text{ED interaction}$$

E_i, $|i\rangle$: energies, energy levels of the atom

Dipole Operator:

$$(2) \quad \boxed{\hat{\vec{\mu}} = \sum_{i,j} \vec{\mu}_{ij} |i\rangle\langle j| = \sum_{i,j} \vec{\mu}_{ij}\hat{\sigma}_{ij}}$$

Field Operator:

$$\hat{\vec{E}}(\vec{r},t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + H.C., \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar\omega_{\vec{k}}}{2\varepsilon_0 V}}$$

2 polarization modes implicit

Pin down atom where $u_{\vec{k}}(\vec{r}) = 1$

– anywhere if $u_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$

– if $u_{\vec{k}}(\vec{r}) = \sin(kz)$ then where $\sin(kz) = 1$

$$(3) \quad \boxed{\hat{\vec{E}}(\vec{r},t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \left(\hat{a}_{\vec{k}} + \hat{a}^+_{\vec{k}}\right)}$$

Quantized Light – Matter Interactions

Dipole Operator:

(2) $$\boxed{\hat{\vec{\mu}} = \sum_{i,j} \vec{\mu}_{ij} |i\rangle\langle j| = \sum_{i,j} \vec{\mu}_{ij} \hat{\sigma}_{ij}}$$

Field Operator:

$$\vec{E}(\vec{r},t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + H.c., \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar\omega_{\vec{k}}}{2\varepsilon_0 V}}$$

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(3) $$\boxed{\hat{\vec{E}}(\vec{r},t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})}$$

Combining (2) & (3):

$$\hat{H}_{AF} = \sum_{ij} \sum_{\vec{k}} -\vec{\mu}_{ij} \cdot \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})$$

$$= \sum_{ij} \sum_{\vec{k}} \hbar g^{(ij)}_{\vec{k}} \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})$$

where $$g^{(ij)}_{\vec{k}} = \frac{\vec{\mu}_{ij} \cdot \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}}}{\hbar}$$

Rabi Freq., note sign convention

2-level atom ⇨ $(i,j) = (1,2)$ **:**

$$\boxed{\hat{H}_{AF} = \hbar \sum_{\vec{k}} (g_{\vec{k}} \hat{\sigma}_{21} + g^{*}_{\vec{k}} \hat{\sigma}_{12})(\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})}$$

Define:

$$\left.\begin{aligned} \hat{\sigma}_{+} &= \hat{\sigma}_{21} = |2\rangle\langle 1| \\ \hat{\sigma}_{-} &= \hat{\sigma}_{12} = |1\rangle\langle 2| \\ \hat{\sigma}_{z} &= \hat{\sigma}_{22} - \hat{\sigma}_{11} = |2\rangle\langle 2| - |1\rangle\langle 1| \end{aligned}\right\}$$

Pauli matrices

$$\hat{\sigma}_x = \frac{1}{2}(\hat{\sigma}_{+} + \hat{\sigma}_{-})$$

$$\hat{\sigma}_y = \frac{1}{2i}(\hat{\sigma}_{+} - \hat{\sigma}_{-})$$

$$\hat{\sigma}_z$$

Quantized Light – Matter Interactions

Combining (2) & (3):

$$\boxed{\begin{aligned}\hat{H}_{AF} &= \sum_{ij}\sum_{\vec{k}} -\vec{r}_{ij}\cdot\vec{\varepsilon}_{\vec{k}}\,\mathcal{E}_{\vec{k}}\,\hat{\sigma}_{ij}\,(\hat{a}_{\vec{k}}+\hat{a}^{+}_{\vec{k}})\\ &= \sum_{ij}\sum_{\vec{k}} \hbar\, g^{(ij)}_{\vec{k}}\,\hat{\sigma}_{ij}\,(\hat{a}_{\vec{k}}+\hat{a}^{+}_{\vec{k}})\\ \text{where}\quad g^{(ij)}_{\vec{k}} &= \frac{\vec{r}_{ij}\cdot\vec{\varepsilon}_{\vec{k}}\,\mathcal{E}_{\vec{k}}}{\hbar}\end{aligned}}$$

Rabi Freq., note sign convention

2-level atom ⇨ $(i,j)=(1,2)$:

$$\boxed{\hat{H}_{AF} = \hbar\sum_{\vec{k}}\left(g_{\vec{k}}\hat{\sigma}_{21}+g^{*}_{\vec{k}}\hat{\sigma}_{12}\right)(\hat{a}_{\vec{k}}+\hat{a}^{+}_{\vec{k}})}$$

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$$\hat{\sigma}_{y} = \frac{1}{2i}(\hat{\sigma}_{+}-\hat{\sigma}_{-})$$

$$\hat{\sigma}_{z}$$

With this notation **(4)**

$$\hat{H}_{AF} = \hbar\sum_{\vec{k}}\left(g_{\vec{k}}\hat{\sigma}_{+}\hat{a}_{\vec{k}} + \cancel{g_{\vec{k}}\hat{\sigma}_{+}\hat{a}^{+}_{\vec{k}}} + \cancel{g_{\vec{k}}\hat{\sigma}_{-}\hat{a}_{\vec{k}}} + g_{\vec{k}}\hat{\sigma}_{-}\hat{a}^{+}_{\vec{k}}\right)$$

Energy conservation?

$$\hat{H}_{AF} = \hbar\sum_{\vec{k}}\left(g_{\vec{k}}\hat{\sigma}_{+}\hat{a}_{\vec{k}} + g_{\vec{k}}\hat{\sigma}_{-}\hat{a}^{+}_{\vec{k}}\right)$$

Putting it all together

$$\boxed{\begin{aligned}&\hat{H} = \hat{H}_{F}+\hat{H}_{A}+\hat{H}_{AF} = \qquad\qquad \textbf{(5)}\\ &\sum_{\vec{k}}\hbar\omega_{\vec{k}}\hat{a}^{+}_{\vec{k}}\hat{a}_{\vec{k}} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_{z} + \sum_{\vec{k}}\hbar\left(g_{\vec{k}}\hat{\sigma}_{+}\hat{a}_{\vec{k}} + g_{\vec{k}}\hat{\sigma}_{-}\hat{a}^{+}_{\vec{k}}\right)\end{aligned}}$$

We changed the zero point for energy by subtracting

$\sum_{\vec{k}}\frac{1}{2}\hbar\omega_{\vec{k}}$ and $\frac{1}{2}(E_2-E_1)$

field atom

Quantized Light – Matter Interactions

Combining (2) & (3):

$$\boxed{\begin{aligned}\hat{H}_{AF} &= \sum_{ij}\sum_{\vec{k}} -\vec{\eta}_{ij}\cdot\vec{\varepsilon}_{\vec{k}}\,\mathcal{E}_{\vec{k}}\,\hat{\sigma}_{ij}\,(\hat{a}_{\vec{k}}+\hat{a}^{+}_{\vec{k}})\\ &= \sum_{ij}\sum_{\vec{k}} \hbar g^{(ij)}_{\vec{k}}\,\hat{\sigma}_{ij}\,(\hat{a}_{\vec{k}}+\hat{a}^{+}_{\vec{k}})\\ \text{where}\quad g^{(ij)}_{\vec{k}} &= \frac{\vec{\eta}_{ij}\cdot\vec{\varepsilon}_{\vec{k}}\,\mathcal{E}_{\vec{k}}}{\hbar}\end{aligned}}$$

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With this notation **(4)**

$$\hat{H}_{AF} = \hbar\sum_{\vec{k}}\left(g_{\vec{k}}\hat{\sigma}_{+}\hat{a}_{\vec{k}} + \cancel{g_{\vec{k}}\hat{\sigma}_{+}\hat{a}^{+}_{\vec{k}}} + \cancel{g_{\vec{k}}\hat{\sigma}_{-}\hat{a}_{\vec{k}}} + g_{\vec{k}}\hat{\sigma}_{-}\hat{a}^{+}_{\vec{k}}\right)$$

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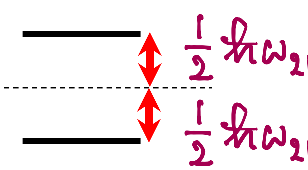
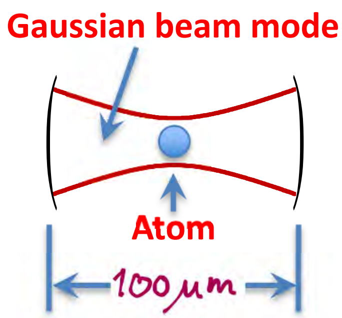


Quantized Light – Matter Interactions

Interaction with Single-mode Fields

Good approx. in small, high-Q Cavity



$c/2L \gg A_{21}$

$|g_{\vec{k}}| \gg A_{21}, \gamma$

Single-mode (Jaynes-Cummings) Hamiltonian

$$\boxed{\hat{H} = \underbrace{\hbar\omega\,\hat{a}^{\dagger}\hat{a} + \tfrac{1}{2}\hbar\omega_{21}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar g(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^{\dagger})}_{H_{AF}}}$$

Foundational result for remainder of the course

Quantized Light – Matter Interactions

More about Single-mode Cavity QED

$$\hat{H} = \underbrace{\hbar\omega\hat{a}^{+}\hat{a} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar(g_{\vec{k}}\hat{\sigma}_{+} + g^{*}_{\vec{k}}\hat{\sigma}_{-})(\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})}_{H_{AF}}$$

For simplicity $\vec{\mu}_{21} = \vec{\mu}_{12} \Rightarrow g_{\vec{k}} = g^{*}_{\vec{k}}$

Note: $\hat{H}_{AF}$ **conserves excitation number,**
couples $|2,n\rangle \leftrightarrow |1,n+1\rangle$

Series of 2-level systems, one for each n

All 2-level systems are alike
Rabi problem!

Switch to Interaction Picture: **Sakurai page 318-319**

$$\left.\begin{aligned}\hat{H}_S \rightarrow \hat{H}_I &= e^{i\frac{\hat{H}_0}{\hbar}t}\hat{H}_{AF}e^{-i\frac{\hat{H}_0}{\hbar}t} \\ |\psi_S(t)\rangle \rightarrow |\psi_I(t)\rangle &= e^{i\frac{\hat{H}_0}{\hbar}t}|\psi_S(t)\rangle\end{aligned}\right\} \Rightarrow$$

Can show

$$e^{i\omega\hat{a}^{+}\hat{a}t}\,\hat{a}\,e^{-i\omega\hat{a}^{+}\hat{a}t} = \hat{a}\,e^{-i\omega t}$$

$$e^{i\frac{\omega_{21}}{2}\hat{\sigma}_z t}\,\hat{\sigma}_{+}\,e^{-i\frac{\omega_{21}}{2}\hat{\sigma}_z t} = \hat{\sigma}_{+}\,e^{-i\omega_{21}t}$$

Quantized Light – Matter Interactions

More about Single-mode Cavity QED

$$\hat{H} = \underbrace{\hbar\omega\,\hat{a}^{+}\hat{a} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_{z}}_{H_0} + \underbrace{\hbar\left(g_{\vec{k}}\hat{\sigma}_{+} + g_{\vec{k}}^{*}\hat{\sigma}_{-}\right)\left(\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^{+}\right)}_{H_{AF}}$$

For simplicity $\vec{\mu}_{21} = \vec{\mu}_{12}$ ⇨ $g_{\vec{k}} = g_{\vec{k}}^{*}$

Note: $\hat{H}_{AF}$ conserves excitation number, couples $|2,n\rangle \leftrightarrow |1,n+1\rangle$

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Switch to Interaction Picture:

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$$\left.\begin{aligned}\hat{H}_S &\rightarrow \hat{H}_I = e^{i\frac{\hat{H}_0}{\hbar}t}\hat{H}_{AF}e^{-i\frac{\hat{H}_0}{\hbar}t} \\ |\psi_S(t)\rangle &\rightarrow |\psi_I(t)\rangle = e^{i\frac{\hat{H}_0}{\hbar}t}|\psi_S(t)\rangle\end{aligned}\right\}$$

$$\hat{H}_{AF} = \hbar g\left(\hat{\sigma}_{+}\hat{a}e^{i(\omega_{21}-\omega)t} + \hat{\sigma}_{+}\hat{a}^{+}e^{i(\omega_{21}+\omega)t} + \hat{\sigma}_{-}\hat{a}e^{-i(\omega_{21}+\omega)t} + \hat{\sigma}_{-}\hat{a}^{+}e^{-i(\omega_{21}-\omega)t}\right)$$

RWA and resonant approximation

Jaynes-Cummings Hamiltonian

$$\hat{H}_I = \hat{H}_{AF} = \hbar g\left(\hat{\sigma}_{+}\hat{a}e^{i\Delta t} + \hat{\sigma}_{-}\hat{a}^{+}e^{-i\Delta t}\right)$$

$\Delta = \omega_{21} - \omega$

Important: Change in notation

For the remainder of this course we change indices 1 to g (ground state) and 2 to e (excited state). Thus a g appearing inside a ket refers to a state, elsewhere g is a Rabi frequency. This is needed for clarity.

Eigenstates of $\hat{H}_0 = \hat{H}_F + \hat{H}_A$

State	Energy
$\|e,n\rangle$	$\hbar\omega n + \frac{1}{2}\hbar\omega_{21}$
$\|g,n+1\rangle$	$\hbar\omega(n+1) - \frac{1}{2}\hbar\omega_{21}$

Quantized Light – Matter Interactions

$$\hat{H}_{AF} = \hbar g \left(\hat{\sigma}_+ \hat{a} e^{i(\omega_{21}-\omega)t} + \hat{\sigma}_+ \hat{a}^+ e^{i(\omega_{21}+\omega)t} + \hat{\sigma}_- \hat{a} e^{-i(\omega_{21}+\omega)t} + \hat{\sigma}_- \hat{a}^+ e^{-i(\omega_{21}-\omega)t} \right)$$

RWA and resonant approximation

Jaynes-Cummings Hamiltonian

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Eigenstates of $\hat{H}_0 = \hat{H}_F + \hat{H}_A$

State	Energy
$\|e,n\rangle$	$\hbar\omega n + \frac{1}{2}\hbar\omega_{21}$
$\|g,n+1\rangle$	$\hbar\omega(n+1) - \frac{1}{2}\hbar\omega_{21}$

Cavity QED version of the Rabi Problem

$$|\psi(0)\rangle = |e,n\rangle$$

$$|\psi(t)\rangle = C_{g,n+1}|g,n+1\rangle + C_{e,n}|e,n\rangle$$

Matrix elements

$$\langle e,n|\hat{H}_{AF}|g,n+1\rangle = \hbar g\sqrt{n+1}\, e^{i\Delta t}$$

$$\langle g,n+1|\hat{H}_{AF}|e,n\rangle = \hbar g\sqrt{n+1}\, e^{-i\Delta t}$$

Schrödinger Equation

$$i\hbar \frac{d}{dt}\begin{pmatrix} C_{g,n+1} \\ C_{e,n} \end{pmatrix} = \hbar g\sqrt{n+1}\begin{pmatrix} 0 & e^{-i\Delta t} \\ e^{i\Delta t} & 0 \end{pmatrix}\begin{pmatrix} C_{g,n+1} \\ C_{e,n} \end{pmatrix}$$

Quantized Light – Matter Interactions

Cavity QED version of the Rabi Problem

$$|\psi(0)\rangle = |e,n\rangle$$

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Matrix elements

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Schrödinger Equation

$$i\hbar\frac{d}{dt}\begin{pmatrix} C_{g,n+1} \\ C_{e,n} \end{pmatrix} =$$

$$\hbar g\sqrt{n+1}\begin{pmatrix} 0 & e^{-i\Delta t} \\ e^{i\Delta t} & 0 \end{pmatrix}\begin{pmatrix} C_{g,n+1} \\ C_{e,n} \end{pmatrix}$$

$$\boxed{\begin{aligned} \dot{C}_{g,n+1} &= -ig\sqrt{n+1}\,e^{-i\Delta t}\,C_{e,n} \\ \dot{C}_{e,n} &= -ig\sqrt{n+1}\,e^{i\Delta t}\,C_{g,n+1} \end{aligned}}$$

Substitute $C_{g,n+1} \rightarrow C_e$, $C_{e,n} \rightarrow C_e e^{i\Delta t}$

Looks exactly like Semiclassical Rabi problem

Solve for $C_g(0) = 0$, $C_e(0) = 1$

$$\boxed{\begin{aligned} C_{e,n}(t) &= \left[\cos\left(\frac{\Omega_n t}{2}\right) - i\frac{\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]e^{i\Delta t/2} \\ C_{g,n+1} &= -i\,\frac{2g\sqrt{n+1}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)e^{-i\Delta t/2} \\ \Omega_n &= \left(4g^2(n+1) + \Delta^2\right)^{1/2} \end{aligned}}$$

Quantized Light – Matter Interactions

$$\dot{c}_{g,n+1} = -ig\sqrt{n+1}\, e^{-i\Delta t}\, c_{e,n}$$

$$\dot{c}_{e,n} = -ig\sqrt{n+1}\, e^{i\Delta t}\, c_{g,n+1}$$

Substitute $c_{g,n+1} \rightarrow c_g\,,\quad c_{e,n} \rightarrow c_e\, e^{i\Delta t}$

Looks exactly like Semiclassical Rabi problem

Solve for $c_g(0) = 0\,,\quad c_e(0) = 1$

$$c_{e,n}(t) = \left[\cos\left(\frac{\Omega_n t}{2}\right) - i\,\frac{\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right] e^{i\Delta t/2}$$

$$c_{e,n+1} = -i\,\frac{2g\sqrt{n+1}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right) e^{-i\Delta t/2}$$

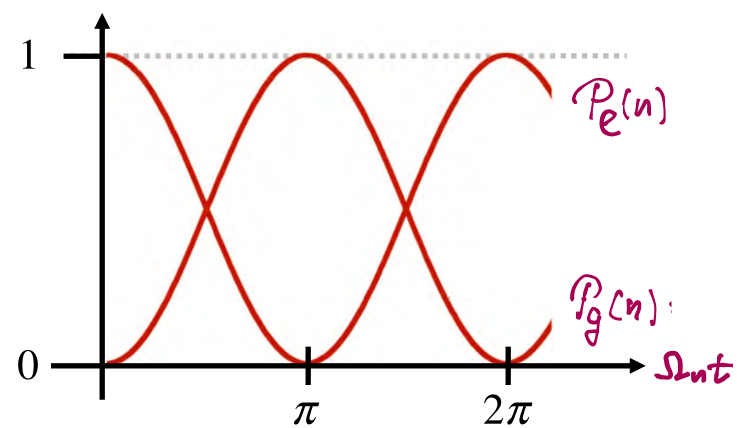
$$\Omega_n = \left(4g^2(n+1) + \Delta^2\right)^{1/2}$$

Rabi Oscillations

$$P_e(n) = \cos^2\left(\frac{\Omega_n t}{2}\right) + \left(\frac{\Delta}{\Omega_n}\right)^2 \sin^2\left(\frac{\Omega_n t}{2}\right)$$

$$P_g(n) = \frac{4g^2(n+1)}{\Omega_n^2}\sin^2\left(\frac{\Omega_n t}{2}\right)$$

Example: $\Delta = 0$



Quantized Light – Matter Interactions

Vacuum Rabi Oscillations

If $|\psi(0)\rangle = |e,0\rangle$ $\Rightarrow$ no photons in field

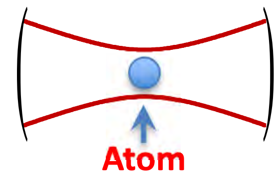
yet $|e,0\rangle$ evolves into $|g,1\rangle$

Uniquely QED phenomenon!

Asymmetry $\begin{cases} |e,n=0\rangle \longrightarrow |g,n=1\rangle \\ |g,n=0\rangle \not\Rightarrow |g,n=1\rangle \end{cases}$

holds germ of **Spontaneous Decay**

Next: More Cavity QED



2-level atom

Single cavity mode

What happens with a Coherent State in the Cavity mode?

(Quantum-Classical correspondence)

Initial atom-field state:

$|\psi(0)\rangle = |1\rangle \otimes |\alpha\rangle$