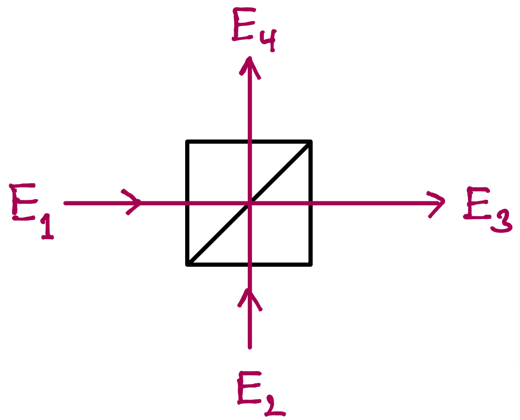
Application: Classical & Quantum Beamsplitters

"Classical Beamsplitter"



Coupled H & V modes

Linear symmetric input-output map

$$E_3 = tE_1 + rE_2$$
$$E_4 = rE_1 + tE_2$$

Energy conservation requires

$$\mathcal{E} = |E_1|^2 + |E_2|^2 = |E_3|^2 + |E_4|^2$$

Total energy input

Choose $E_1 = \mathcal{E}$, $E_2 = 0$

$$\mathcal{E} = |E_3|^2 + |E_4|^2 = \mathcal{E}\left(|t|^2 + |r|^2\right)$$

Choose $E_1 = \frac{1}{\sqrt{2}}\mathcal{E}$, $E_2 = \frac{1}{\sqrt{2}}\mathcal{E}$

$$\mathcal{E} = |E_3|^2 + |E_4|^2 = \frac{1}{2}\mathcal{E}\,|t+r|^2$$

$$|t|^2 + |r|^2 + tr^* + rt^* = 1$$

From this it follows that

$$\boxed{\begin{gathered} |t|^2 + |r|^2 = 1 \\ tr^* + rt^* = 0 \end{gathered}}$$

Classical input-output map

$$\boxed{\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}}$$

Quantum Beamsplitter

Heisenberg Picture

Field Operators obey Maxwells Eqs

Classical field

$$E_\perp(\vec{r}, t) \propto \alpha(t)$$

Quantum equivalent

$$\hat{E}^{(+)}_\perp(\vec{r}, t) \propto \hat{a}(t)$$

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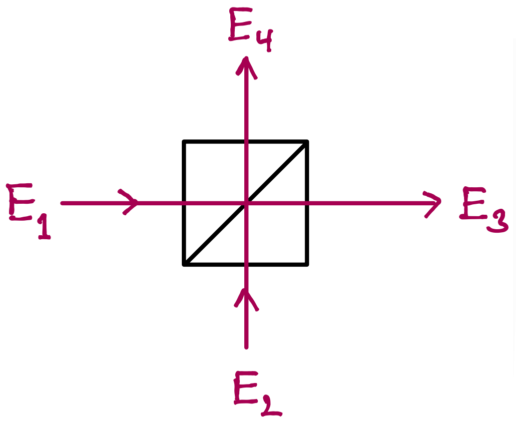
“Classical Beamsplitter”



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Quantum Beamsplitter

Heisenberg Picture ➡ **Field Operators obey Maxwells Eqs**

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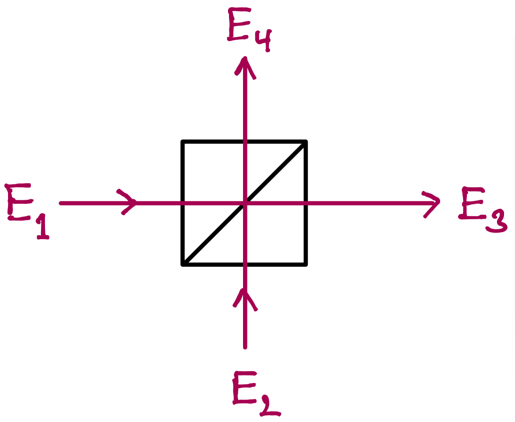
"Classical Beamsplitter"



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Total energy input

Choose $E_1 = \mathcal{E}\ ,\ E_2 = 0$ ⇨

$$\mathcal{E} = |E_3|^2 + |E_4|^2 = \mathcal{E}\left(|t|^2 + |r|^2\right)$$

Choose $E_1 = \frac{1}{\sqrt{2}}\mathcal{E}\ ,\ E_2 = \frac{1}{\sqrt{2}}\mathcal{E}$ ⇨

$$\mathcal{E} = |E_3|^2 + |E_4|^2 = \frac{1}{2}\mathcal{E}\,|t + r|^2$$ ⇨

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Quantum Beamsplitter

Heisenberg Picture ⇨ **Field Operators obey Maxwells Eqs**

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Quantum Beamsplitter

Heisenberg Picture → Field Operators obey Maxwells Eqs

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Quantum Beamsplitter

$$\begin{pmatrix}\hat{E}_3\\\hat{E}_4\end{pmatrix}=\begin{pmatrix}t & r\\ r & t\end{pmatrix}\begin{pmatrix}\hat{E}_1\\\hat{E}_2\end{pmatrix}$$

Quantum input-output map

$$\boxed{\begin{pmatrix}\hat{a}_3\\\hat{a}_4\end{pmatrix}=\begin{pmatrix}t & r\\ r & t\end{pmatrix}\begin{pmatrix}\hat{a}_1\\\hat{a}_2\end{pmatrix}}$$

Invert Map

$$\begin{gathered}\hat{a}_3=t\hat{a}_1+r\hat{a}_2\\ \hat{a}_4=r\hat{a}_1+t\hat{a}_2\end{gathered}\;\Rightarrow\;\begin{gathered}\hat{a}_1=t^*\hat{a}_3+r^*\hat{a}_4\\ \hat{a}_2=r^*\hat{a}_3+t^*\hat{a}_4\end{gathered}$$

Switch to creation operators →

$$\boxed{\begin{gathered}\hat{a}_1^\dagger=t\hat{a}_3^\dagger+r\hat{a}_4^\dagger\\ \hat{a}_2^\dagger=r\hat{a}_3^\dagger+t\hat{a}_4^\dagger\end{gathered}}$$

Application: Classical & Quantum Beamsplitters

Quantum Beamsplitter

$$\begin{pmatrix} \hat{E}_3 \\ \hat{E}_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \hat{E}_1 \\ \hat{E}_2 \end{pmatrix}$$

Quantum input-output map

$$\boxed{\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}}$$

Invert Map

$$\hat{a}_3 = t\hat{a}_1 + r\hat{a}_2 \qquad \hat{a}_1 = t^*\hat{a}_3 + r^*\hat{a}_4$$

$$\hat{a}_4 = r\hat{a}_1 + t\hat{a}_2 \qquad \hat{a}_2 = r^*\hat{a}_3 + t^*\hat{a}_4$$

Switch to creation operators

$$\boxed{\begin{aligned} \hat{a}_1^\dagger &= t\hat{a}_3^\dagger + r\hat{a}_4^\dagger \\ \hat{a}_2^\dagger &= r\hat{a}_3^\dagger + t\hat{a}_4^\dagger \end{aligned}}$$

Switch to Schrödinger Picture

General input state: 2-mode vacuum

$$\boxed{|\psi_{in}\rangle = \sum_{nm} f_n \frac{1}{\sqrt{n!}} (\hat{a}_1^\dagger)^n g_m \frac{1}{\sqrt{m!}} (\hat{a}_2^\dagger)^m |0\rangle}$$

The BS maps $\hat{a}_1^\dagger, \hat{a}_2^\dagger$ to linear combinations of $\hat{a}_3^\dagger, \hat{a}_4^\dagger$

General output state: (Schrödinger Picture)

$$\boxed{|\psi_{out}\rangle = \sum_{nm} f_n \frac{1}{\sqrt{n!}} (t\hat{a}_3^\dagger + r\hat{a}_4^\dagger)^n g_m \frac{1}{\sqrt{m!}} (r\hat{a}_3 + t\hat{a}_4)^m |0\rangle}$$

Example: One-photon input state

$$|\psi_{in}\rangle = |1\rangle_1 |0\rangle_2 = \hat{a}_1^\dagger |0\rangle$$

$$|\psi_{out}\rangle = (t\hat{a}_3^\dagger + r\hat{a}_4^\dagger)|0\rangle = t|1\rangle_3 |0\rangle_4 + r|0\rangle_3 |1\rangle_4$$

Application: Classical & Quantum Beamsplitters

Switch to Schrödinger Picture

General input state: 2-mode vacuum

$$\boxed{|\psi_{in}\rangle = \sum_{nm} f_n \frac{1}{\sqrt{n!}}(\hat{a}_1^+)^n g_m \frac{1}{\sqrt{m!}}(\hat{a}_2^+)^m |0\rangle}$$

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General output state: (Schrödinger Picture)

$$\boxed{|\psi_{out}\rangle = \sum_{nm} f_n \frac{1}{\sqrt{n!}}(t\hat{a}_3^+ + r\hat{a}_4^+)^n g_m \frac{1}{\sqrt{m!}}(r\hat{a}_3 + t\hat{a}_4)^m |0\rangle}$$

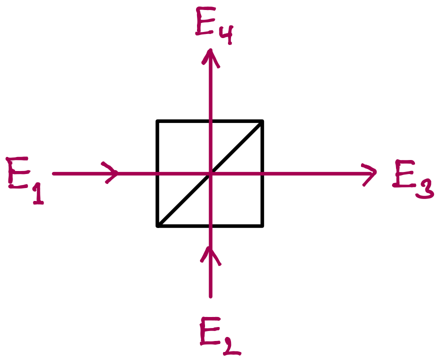
Example: One-photon input state

$$|\psi_{in}\rangle = |1\rangle_1 |0\rangle_2 = \hat{a}_1^+|0\rangle$$

$$|\psi_{out}\rangle = (t\hat{a}_3^+ + r\hat{a}_4^+)|0\rangle = t|1\rangle_3|0\rangle_4 + r|0\rangle_3|1\rangle_4$$

50/50 Beamsplitter

$t = 1/\sqrt{2}, \ r = i/\sqrt{2}$



$$\boxed{|\psi_{out}\rangle = \frac{1}{\sqrt{2}}(|1\rangle_3|0\rangle_4 + i|0\rangle_3|1\rangle_4)}$$

Note: This is a Photon number-Mode Entangled State

(∗) A coherent superposition of states w/ one photon in port 3 and zero in port 4, and zero in port 3 and one in port 4.

Can we assign states such as, e. g.

$\frac{1}{\sqrt{2}}(|1\rangle_3 + i|0\rangle_3)$ to port 3

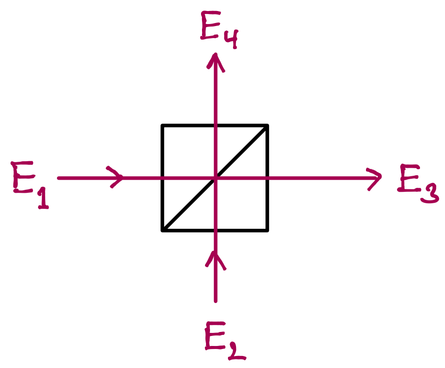
?

$\frac{1}{\sqrt{2}}(|0\rangle_4 + i|1\rangle_4)$ to port 4

Viewed on their own, each port is in a mixed state

Application: Classical & Quantum Beamsplitters

50/50 Beamsplitter



$$t = 1/\sqrt{2},\ r = i/\sqrt{2}$$

$$\downarrow$$

$$\boxed{|\psi_{out}\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle_3|0\rangle_4 + i\,|0\rangle_3|1\rangle_4\right)}$$

Note: This is a Photon number-Mode Entangled State

(∗) A coherent superposition of states w/ one photon in port 3 and zero in port 4, and zero in port 3 and one in port 4.

Can we assign states such as, e. g.

$$\xcancel{\frac{1}{\sqrt{2}}\left(|1\rangle_3 + i\,|0\rangle_3\right)} \text{ to port 3}$$

$$\xcancel{\frac{1}{\sqrt{2}}\left(|0\rangle_4 + i\,|1\rangle_4\right)} \text{ to port 4}$$

!

Viewed on their own, each port is in a <u>mixed state</u>

Example: Two-photon input state, 50/50 BS

$$|\psi_{in}\rangle = \hat{a}_1^+\hat{a}_2^+|0\rangle$$

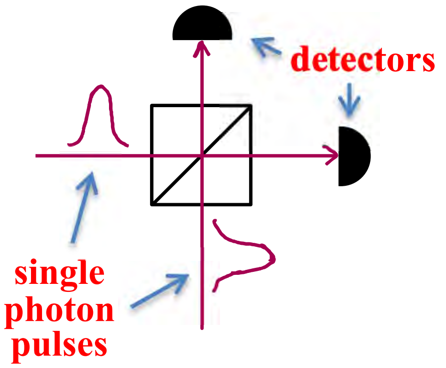
$$|\psi_{out}\rangle = \frac{1}{2}(\hat{a}_3^+ + i\hat{a}_4^+)(i\hat{a}_3^+ + \hat{a}_4^+)|0\rangle$$

destructive interference

$$= \frac{1}{2}(i\hat{a}_3^+\hat{a}_3^+ + i\hat{a}_4^+\hat{a}_4^+ + \hat{a}_3^+\hat{a}_4^+ - \hat{a}_4^+\hat{a}_3^+)|0\rangle$$

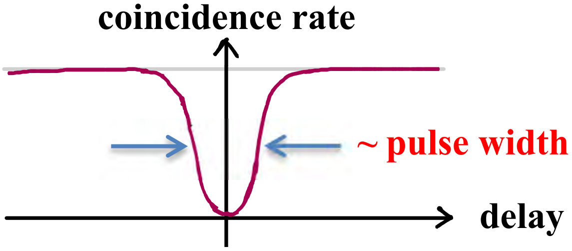
$$= \frac{i}{2}(\hat{a}_3^+\hat{a}_3^+ + \hat{a}_4^+\hat{a}_4^+)|0\rangle = \frac{i}{\sqrt{2}}\left(|2\rangle_3|0\rangle_4 + |0\rangle_3|2\rangle_4\right)$$

Experiment:

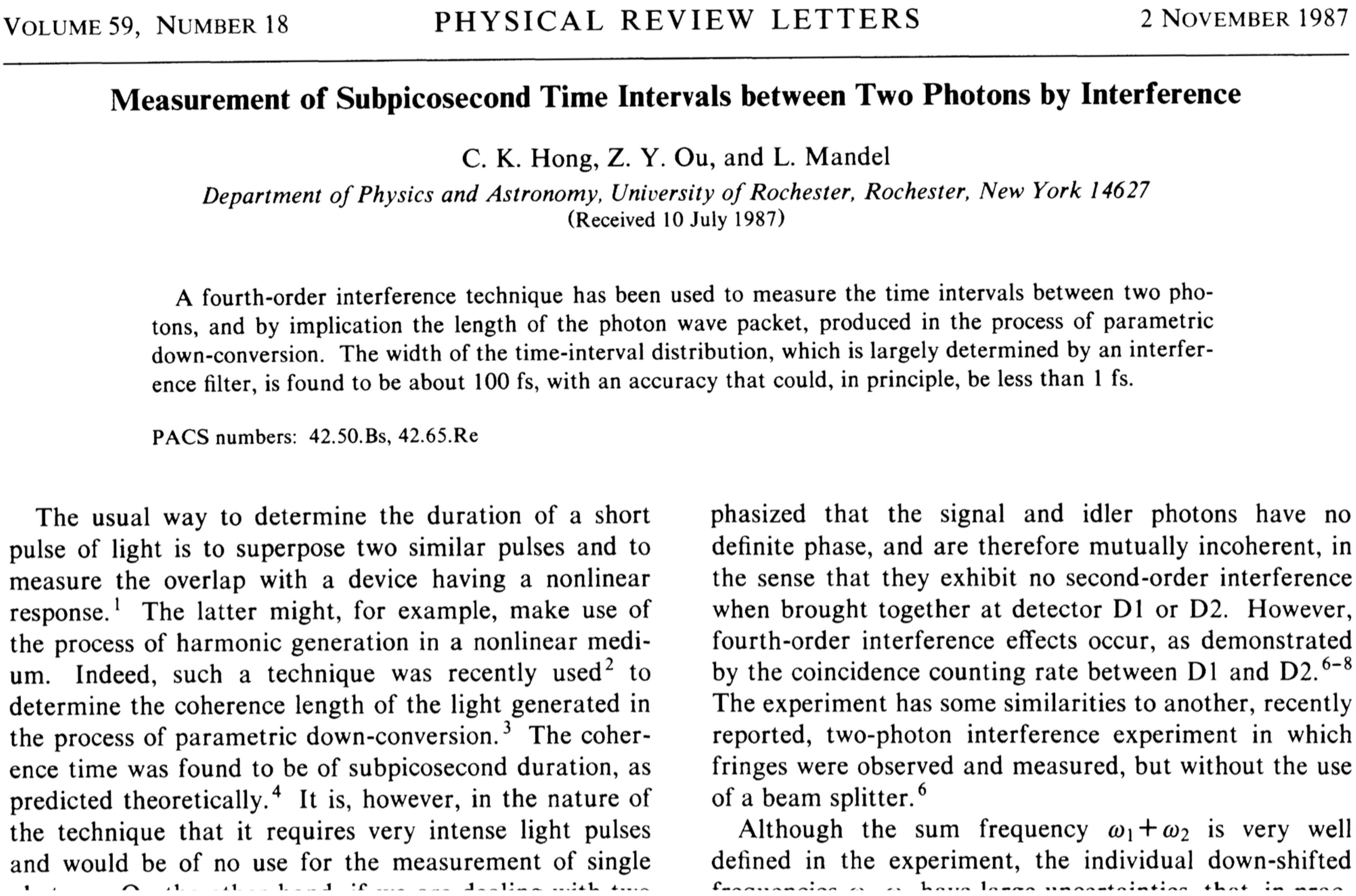


Coincidence detections are never seen when pulses overlap -> “bunching”.

Delay between pulses leads to Coincidence detections.



Application: Classical & Quantum Beamsplitters



VOLUME 59, NUMBER 18 PHYSICAL REVIEW LETTERS 2 NOVEMBER 1987

Measurement of Subpicosecond Time Intervals between Two Photons by Interference

C. K. Hong, Z. Y. Ou, and L. Mandel
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

(Received 10 July 1987)

A fourth-order interference technique has been used to measure the time intervals between two photons, and by implication the length of the photon wave packet, produced in the process of parametric down-conversion. The width of the time-interval distribution, which is largely determined by an interference filter, is found to be about 100 fs, with an accuracy that could, in principle, be less than 1 fs.

PACS numbers: 42.50.Bs, 42.65.Re

The usual way to determine the duration of a short pulse of light is to superpose two similar pulses and to measure the overlap with a device having a nonlinear response.[1] The latter might, for example, make use of the process of harmonic generation in a nonlinear medium. Indeed, such a technique was recently used[2] to determine the coherence length of the light generated in the process of parametric down-conversion.[3] The coherence time was found to be of subpicosecond duration, as predicted theoretically.[4] It is, however, in the nature of the technique that it requires very intense light pulses and would be of no use for the measurement of single

phasized that the signal and idler photons have no definite phase, and are therefore mutually incoherent, in the sense that they exhibit no second-order interference when brought together at detector D1 or D2. However, fourth-order interference effects occur, as demonstrated by the coincidence counting rate between D1 and D2.[6-8] The experiment has some similarities to another, recently reported, two-photon interference experiment in which fringes were observed and measured, but without the use of a beam splitter.[6]

Although the sum frequency $\omega_1+\omega_2$ is very well defined in the experiment, the individual down-shifted

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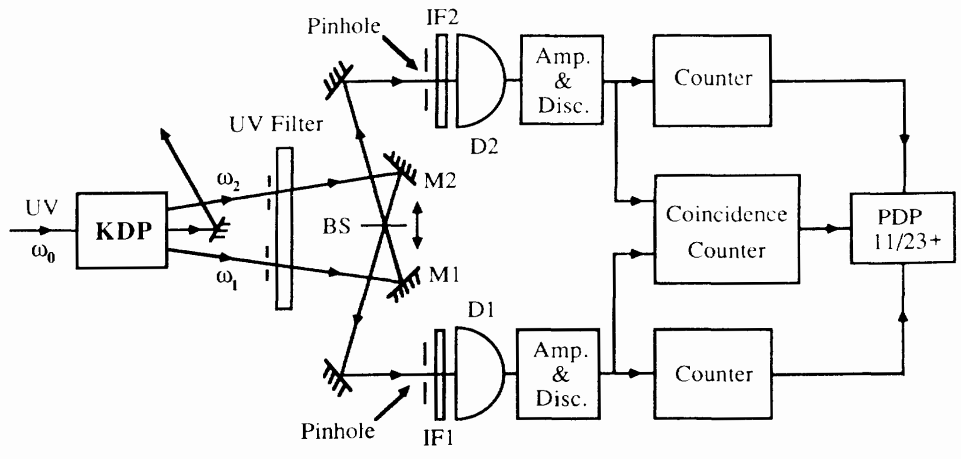
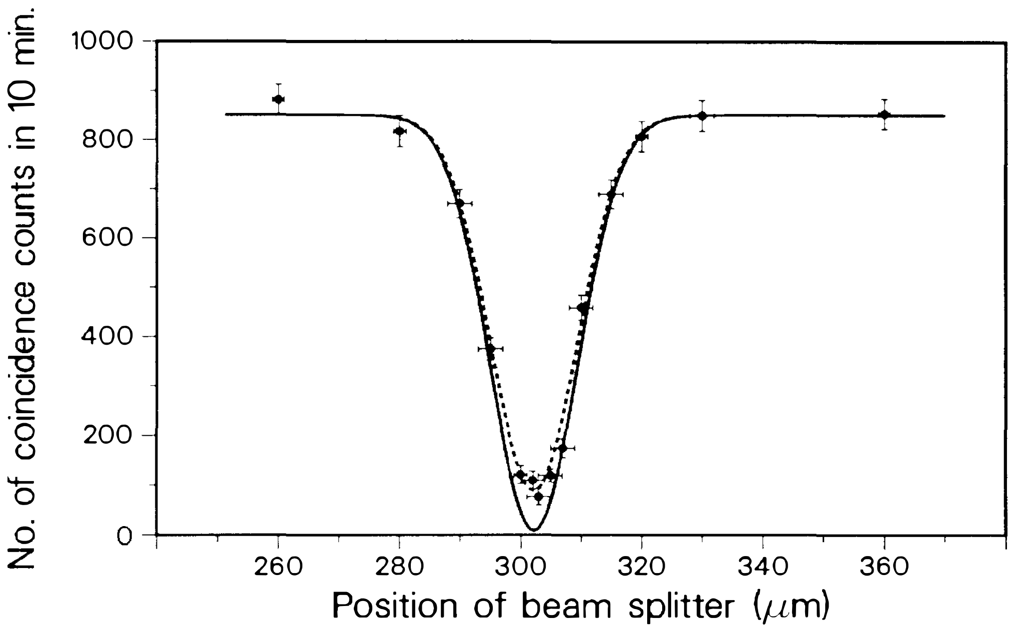


FIG. 1. Outline of the experimental setup.



Quantum States of the Quantized Field

Amplitude and Phase

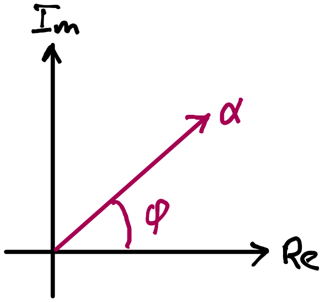
- Key characteristics of classical fields
- Need equivalents for quantum fields

Classical Field

$$E(z,t) = \mathcal{E}_k \boxed{\alpha}\, e^{-i(\omega t - kz)} + c.c.$$

$$\uparrow$$

$$|\alpha| e^{i\varphi}$$



Quantum Field

$$\hat{E}(z,t) = \mathcal{E}_k \boxed{\hat{a}}\, e^{-i(\omega t - kz)} + H.C.$$

Non-Hermitian!

Separate in amplitude & phase?

Consider operators

$$\boxed{\begin{aligned} \hat{a} &= (\hat{N}+1)^{1/2}\, \hat{\exp}(i\varphi) \\ \hat{a}^{+} &= \hat{\exp}(-i\varphi)\,(\hat{N}+1)^{1/2} \end{aligned}} \Rightarrow \boxed{\begin{aligned} \hat{\exp}(i\varphi) &= (\hat{N}+1)^{-1/2}\,\hat{a} \\ \hat{\exp}(-i\varphi) &= \hat{a}^{+}(\hat{N}+1)^{-1/2} \end{aligned}}$$

“phase” **“amplitude”**

“Phase operators”

$$\hat{\exp}(i\varphi)\,\hat{\exp}(-i\varphi) = 1 \qquad \hat{\exp}(i\varphi) = \hat{\exp}(-i\varphi)^{+}$$

$$\hat{\exp}(-i\varphi)\,\hat{\exp}(i\varphi) = 1 \qquad = \left[\hat{\exp}(-i\varphi)\right]^{-1}$$

- Analogous to classical phases
- Non-Hermitian, NOT observables

Quadrature operators?

$$\boxed{\begin{aligned} \hat{\cos}\varphi &= \frac{1}{2}\left[\hat{\exp}(i\varphi) + \hat{\exp}(-i\varphi)\right] \\ &= \frac{1}{2}\left[(\hat{N}+1)^{-1/2}\hat{a} + \hat{a}^{+}(\hat{N}+1)^{-1/2}\right] \\ \hat{\sin}\varphi &= \frac{1}{2i}\left[\hat{\exp}(i\varphi) - \hat{\exp}(i\varphi)\right] \\ &= \frac{1}{2i}\left[(\hat{N}+1)^{-1/2}\hat{a} - \hat{a}^{+}(\hat{N}+1)^{-1/2}\right] \end{aligned}}$$

- Hermitian -> observables
- but ultimately too cumbersome

Let’s rewind and try again…

Quantum States of the Quantized Field

"Phase operators"

$$\hat{\exp}(i\varphi)\hat{\exp}(-i\varphi) = 1 \qquad \hat{\exp}(i\varphi) = \hat{\exp}(-i\varphi)^{+}$$

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$$= \frac{1}{2i}\left[(\hat{N}+1)^{-1/2}\hat{a} - \hat{a}^{+}(\hat{N}+1)^{-1/2}\right]$$

- Hermitian -> observables
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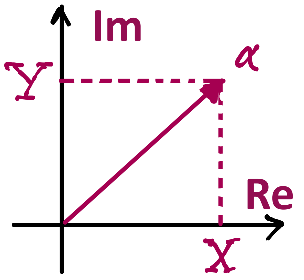
Let's rewind and try again…

Quadratures of the Classical Field – Take Two

$$E(z,t) = \underbrace{\mathcal{E}_k \alpha_k(t)}_{\uparrow} e^{ikz} + c.c.$$

complex amplitude for mode e^{ikz}



Define

$$X(t) = \mathrm{Re}\left[\alpha_k(t)\right] = \frac{1}{2}\left[\alpha_k(t) + \alpha_k^*(t)\right] = Q(t)$$

$$Y(t) = \mathrm{Im}\left[\alpha_k(t)\right] = \frac{1}{2i}\left[\alpha_k(t) - \alpha_k^*(t)\right] = P(t)$$

Quantization: $\alpha \to \hat{a}, \quad \alpha^* \to \hat{a}^{+}$

$$\left.\begin{aligned}\hat{X}(t) &= \frac{1}{2}\left[\hat{a}_k(t) + \hat{a}_k^{+}(t)\right] = \hat{Q}(t)\\ \hat{Y}(t) &= \frac{1}{2i}\left[\hat{a}_k(t) - \hat{a}_k^{+}(t)\right] = \hat{P}(t)\end{aligned}\right\} \quad \left[\hat{X}(t), \hat{Y}(t)\right] = i/2$$

$$\hat{E}(z,t) = \mathcal{E}_k\left(\hat{X}(t) + i\hat{Y}(t)\right)e^{ikz} + H.C.$$

$$= \mathcal{E}_k\left[\hat{X}(t)\cos(kz) - \hat{Y}(t)\sin(kz)\right]$$

– same info, easier to work with –

Quantum States of the Quantized Field

Quadratures of the Classical Field – Take Two

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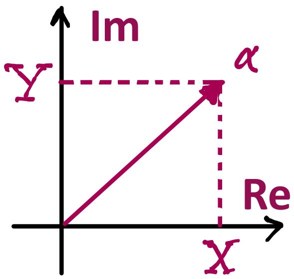

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Quantization: $\alpha \to \hat{a},\ \alpha^* \to \hat{a}^+$

$$\boxed{\begin{aligned} &\left.\begin{aligned} \hat{X}(t) &= \frac{1}{2}[\hat{a}_k(t)+\hat{a}_k^+(t)] = \hat{Q}(t) \\ \hat{Y}(t) &= \frac{1}{2i}[\hat{a}_k(t)-\hat{a}_k^+(t)] = \hat{P}(t) \end{aligned}\right\} \quad [\hat{X}(t),\hat{Y}(t)] = i/2 \\ &\hat{E}(z,t) = \mathcal{E}_k(\hat{X}(t)+i\hat{Y}(t))e^{ikz} + H.C. \\ &\qquad = \mathcal{E}_k[\hat{X}(t)\cos(kz) - \hat{Y}(t)\sin(kz)] \end{aligned}}$$

– same info, easier to work with –

Quantum States of the Field in Mode k

Number States (Foch states)

$$\boxed{\hat{a}^+\hat{a}\,|n\rangle = n\,|n\rangle}$$

$$\langle n|\hat{X}|n\rangle = \langle n|\hat{Y}|n\rangle = 0$$

$$\langle n|\hat{X}^2|n\rangle = \langle n|\hat{Y}^2|n\rangle = \frac{1}{2}(n+1/2)$$

$$\boxed{\Delta X \Delta Y = \frac{1}{2}(n+1/2)}$$

- – HIGHLY non-classical, $\langle \hat{E} \rangle = 0$
- – VERY hard to make for large n

Quantum States of the Quantized Field

Quantum States of the Field in Mode k

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Coherent States (Quasi-classical states)

- Closest approximation to classical field
- See Cohen-Tannoudj, complement G_V

Definition: $|\psi\rangle$ is coherent (quasiclassical) iff

$$\langle\hat{X}(t)\rangle = \langle\psi|\hat{X}(t)|\psi\rangle = X(t), \quad \langle\hat{Y}(t)\rangle = Y(t)$$

$$\langle\hat{H}(t)\rangle = \hbar\omega(|\alpha(t)|^2 + 1/2)$$

noting that

$$\hat{X}(t) \propto \hat{a}(t) = \hat{a}(0)e^{-i\omega t}$$

$$\hat{Y}(t) \propto \hat{a}^+(t) = \hat{a}^+(0)e^{i\omega t}$$

equivalently

Definition: $|\psi\rangle$ is coherent (quasiclassical) iff

(1) $\langle\hat{a}(0)\rangle = \langle\psi|\hat{a}(0)|\psi\rangle = \alpha(0)$

(2) $\langle\hat{a}^+(0)\hat{a}(0)\rangle = \alpha(0)^*\alpha(0)$

Quantum States of the Quantized Field

Coherent States (Quasi-classical states)

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- See Cohen-Tannoudji, complement G_V

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$$\langle \hat{H}(t)\rangle = \hbar\omega\left(|\alpha(t)|^2 + 1/2\right)$$

noting that

$$\hat{X}(t) \propto \hat{a}(t) = \hat{a}(0)e^{-i\omega t}$$

$$\hat{Y}(t) \propto \hat{a}^+(t) = \hat{a}^+(0)e^{i\omega t}$$

equivalently

Definition: $|\psi\rangle$ is coherent (quasiclassical) iff

(1) $\langle \hat{a}(0)\rangle = \langle \psi|\hat{a}(0)|\psi\rangle = \alpha(0)$

(2) $\langle \hat{a}^+(0)\hat{a}(0)\rangle = \alpha(0)^*\alpha(0)$

Cohen-Tannoudji, Lecture Notes

equivalently

Definition: a state $|\alpha\rangle$ is coherent iff

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

Finally, one can show

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

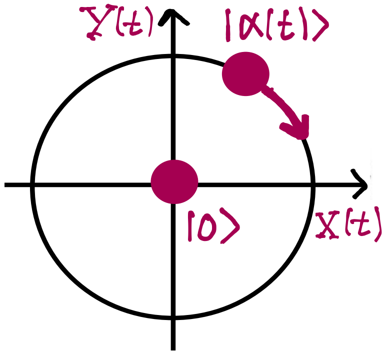
Physical properties

$$\langle \hat{X}(t)\rangle = \mathrm{Re}\left[\alpha(0)e^{-i\omega t}\right]$$

$$\langle \hat{Y}(t)\rangle = \mathrm{Im}\left[\alpha(0)e^{-i\omega t}\right]$$

$$\Delta X(t) = \Delta Y(t) = 1/2$$

$$\Delta X\,\Delta Y = 1/4$$



Quantum States of the Quantized Field

Cohen-Tannoudji, Lecture Notes

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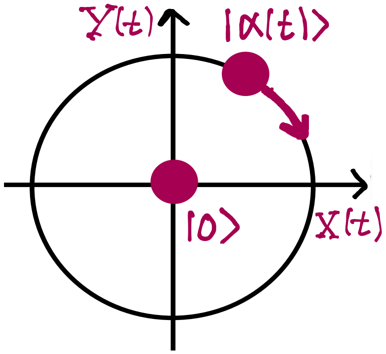

Photon statistics

Measure $\hat{N}$ ⇨ outcomes n

$$P(n) = \langle\alpha|n\rangle\langle n|\alpha\rangle = \frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}$$

Poisson distribution w/
- mean $\bar{n} = |\alpha|^2$
- variance $\Delta n^2 = |\alpha|^2$

$$\Delta n = \sqrt{\bar{n}}$$ – **Shot Noise**

Quantum States of the Quantized Field

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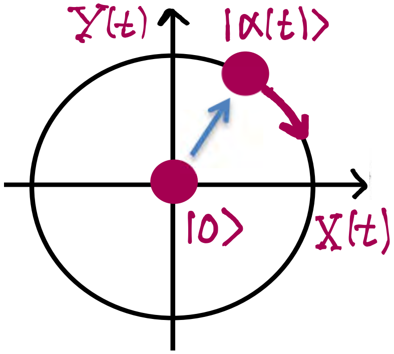
More about Coherent States



⇨ **Coherent States as translated Vacuum States?**

Generating Coherent States from the Vacuum

Definition: $\hat{D}(\alpha) = e^{\alpha\hat{a}^{+} - \alpha^{*}\hat{a}}$

Quantum States of the Quantized Field

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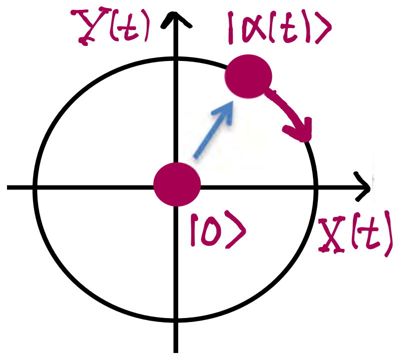

Coherent States as translated Vacuum States?

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Definition: $\hat{D}(\alpha) = e^{\alpha\hat{a}^{+} - \alpha^{*}\hat{a}}$

Unitary, equals translation

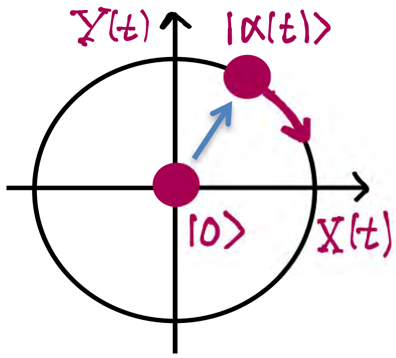
Glaubers formula (from BCH formula)

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{1}{2}[\hat{A},\hat{B}]}$$

for $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$

Quantum States of the Quantized Field

More about Coherent States



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$$\boxed{\hat{D}(\alpha) = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^{+}} e^{-\alpha^{*}\hat{a}}}$$

Remember: $\hat{a}|0\rangle = 0$

$$e^{-\alpha^{*}\hat{a}}|0\rangle = \sum_n \frac{(-\alpha^{*}\hat{a})^n}{n!}|0\rangle = |0\rangle$$

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Quantum States of the Quantized Field

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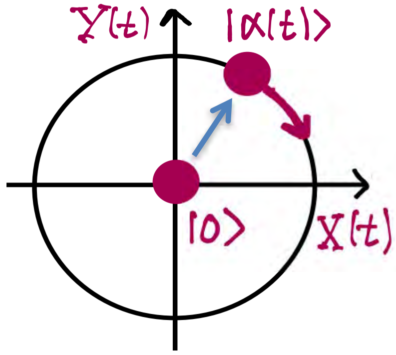

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Quantum States of the Quantized Field

Apply to $$[\underset{\overset{\uparrow}{\hat A}}{\alpha\hat a^+}, \underset{\overset{\uparrow}{\hat B}}{-\alpha^*\hat a}] = \underset{\overset{\uparrow}{[\hat A, \hat B]}}{\alpha^*\alpha}$$

⇨

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$$\boxed{\hat D(\alpha)|0\rangle = |\alpha\rangle}$$

OK – $\hat D(\alpha)$ generates $|\alpha\rangle$ from the vacuum!

Rewrite:

$$\alpha\hat a^+ - \alpha^*\hat a = (\alpha-\alpha^*)\hat X + i(\alpha+\alpha^*)\hat Y$$

$$= i2Y\hat X + i2X\hat Y$$

where $X = \langle\alpha|\hat X|\alpha\rangle$, $Y = \langle\alpha|\hat Y|\alpha\rangle$

Glaubers formula again:

$$\boxed{\hat D(\alpha) = e^{i2Y\hat X + i2X\hat Y} = e^{-XY/4} e^{i2Y\hat X} e^{i2X\hat Y}}$$

Recall: $\hat S(q) = e^{-iq\hat p/\hbar}$ ⇨ **translation by** q

$\hat S(p) = e^{-ip\hat q/\hbar}$ ⇨ **translation by** p

where $q = q_0 X$, $p = p_0 Y$

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Quantum States of the Quantized Field

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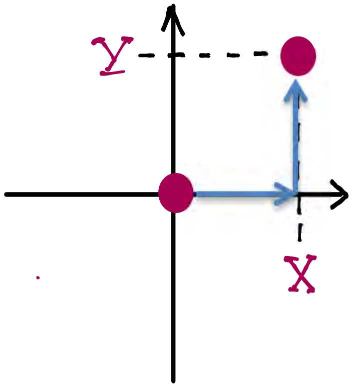
$\hat{q} = q_0\hat{X}$, $\hat{p} = p_0\hat{Y}$ & $q_0 p_0 = 2\hbar$

This gives us

$$\boxed{\hat{S}(q) = \hat{S}(X) = e^{i2X\hat{Y}}, \quad \hat{S}(p) = \hat{S}(Y) = e^{i2Y\hat{X}}}$$

$\hat{D}(\alpha)$ translates along X then Y



Discussion – How to do this?

Quantum States of the Quantized Field

Coherent States from Classical Dipole Radiation

Classical Dipole $d(t) = d_0 \cos(\omega t)$ @ $t = 0$

Quantized Field $\hat{E}(z) = \mathcal{E}_k (\hat{a} + \hat{a}^+)$

Dipole-Field Interaction

$$\hat{H} = \hbar\omega(\hat{a}^+\hat{a} + 1/2) + \hbar\lambda(t)(\hat{a} + \hat{a}^+)$$

$$\lambda(t) = -\frac{d(t)\mathcal{E}_k}{\hbar} = \lambda_0 \cos(\omega t)$$

Drive from $t = 0$ to T

$$\alpha(T) = -i\frac{\lambda_0}{2} e^{-i(\omega-\omega')T/2} \frac{\sin[(\omega-\omega')T/2]}{(\omega-\omega')/2}$$

Recall from Semi-Classical Laser Theory

$\langle \hat{p}(t) \rangle$ drives $\hat{E}(t)$

classical dipole + quantum fluctuations

coherent state + quantum fluctuations

For $t > T$ we have a coherent state

$$\alpha(t) = \alpha(T) e^{-i\omega(t-T)}$$

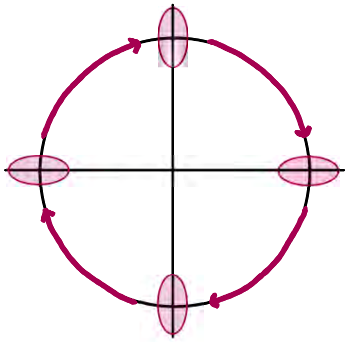
Quantum States of the Quantized Field

Squeezed States

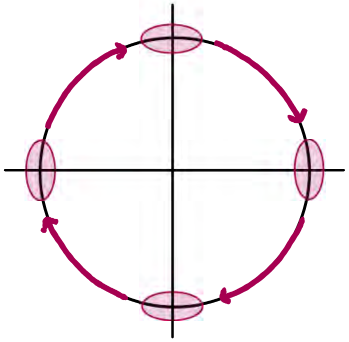
Minimum uncertainty states w/assymmetry

$$\Delta X \Delta Y = 1/4, \quad \Delta X(t) \neq \Delta Y(t)$$

Phase Squeezing



Amplitude Squeezing



Requires interaction with Nonlinear medium

Odds and Ends – Thermal States

$$Z = \mathrm{Tr}\left[e^{-\hat{H}/k_BT}\right]$$

$$\hat{\rho} = \sum_n P(n)|n\rangle\langle n| = \frac{1}{Z}\sum_n e^{-E_n/k_BT}|n\rangle\langle n|$$

$$= (1-q)\sum_n q^n|n\rangle\langle n|, \quad q = e^{-\hbar\omega/k_BT}$$

Mean Photon Number:

$$\bar{n} = \mathrm{Tr}(\hat{\rho}\hat{N}) = \sum_{k,n}\langle k|(1-q)q^n|n\rangle\langle n|\hat{N}|k\rangle$$

$$= (1-q)\sum_n n q^n = \frac{q}{1-q}$$

Photon Number Uncertainty:

$$\langle\hat{N}^2\rangle = (1-q)\sum_n n^2 q^n = \frac{q^2+q}{(1-q)}$$

Quantum States of the Quantized Field

Odds and Ends – Thermal States

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$$\Delta n^2 = \langle\hat{N}^2\rangle - \langle\hat{N}\rangle^2$$

$$= \frac{q^2+q}{(1-q)} - \frac{q^2}{(1-q)^2} = \frac{q}{(1-q)^2}$$

$$\bar{n} = \frac{q}{1-q}$$

Coherent State limit

$$\Delta n = \frac{\sqrt{q}}{1-q} = \sqrt{\bar{n}(\bar{n}+1)} \geq \sqrt{\bar{n}}$$

Optical Frequencies, Room Temperature:

$\lambda = 1\,\mu m$, $\quad T = 300\,K$

$q = 6.5\times10^{-6}$, $\quad \bar{n} \sim 10^{-6}$

Quantum States of the Quantized Field

Odds and Ends – Quantum-Classical Correspondence

Define a Translation Operator

$$\hat{T}_\alpha(t) = e^{\alpha^* e^{i\omega t}\hat{a} - \alpha e^{-i\omega t}\hat{a}^+} = \hat{D}(-\alpha e^{-i\omega t})$$

Use $[\hat{a}, \hat{F}(\hat{a}^+)] = dF(\hat{a}^+)/d\hat{a}^+$ **to show**

$$[\hat{a}, \hat{T}_\alpha] = \hat{a}\hat{T}_\alpha - \hat{T}_\alpha\hat{a} = -\alpha e^{-i\omega t} T_\alpha$$

⇨ $\hat{T}_\alpha \hat{a} = \hat{a}\hat{T}_\alpha + \alpha e^{-i\omega t}\hat{T}_\alpha$

⇨ $\hat{T}_\alpha \hat{a} \hat{T}_\alpha^+ = \hat{a} + \alpha e^{-i\omega t}$

From this we get

(1) Field Observable

$$\hat{E}'_\perp = \hat{T}_\alpha \hat{E}_\perp \hat{T}_\alpha^+ = \hat{T}_\alpha\left(\mathcal{E}_k \hat{a} e^{i\vec{k}\cdot\vec{r}} + H.C.\right)\hat{T}_\alpha^+$$

$$= \overbrace{\mathcal{E}_k \hat{a} e^{i\vec{k}\cdot\vec{r}}} + H.C. + \mathcal{E}_k \alpha e^{-i(\omega t - \vec{k}\cdot\vec{r})} + C.C.$$

$$= \hat{E}_\perp + E_\perp^{Cl}(\alpha, t)$$

(2) Classical Field

(3) Field Observable

We also have $|\psi'(t)\rangle = \hat{T}_\alpha |\alpha(t)\rangle = |0\rangle$

Action of the unitary transformation $\hat{T}_\alpha(t)$

$$\hat{E}'_\perp = \hat{T}_\alpha(t)\,\hat{E}_\perp\,\hat{T}_\alpha(t)^+ = \hat{E}_\perp + E_\perp^{Cl}(\alpha, t)$$

$$|\psi'(t)\rangle = T_\alpha(t)\,|\alpha(t)\rangle = |0\rangle$$

We can work with

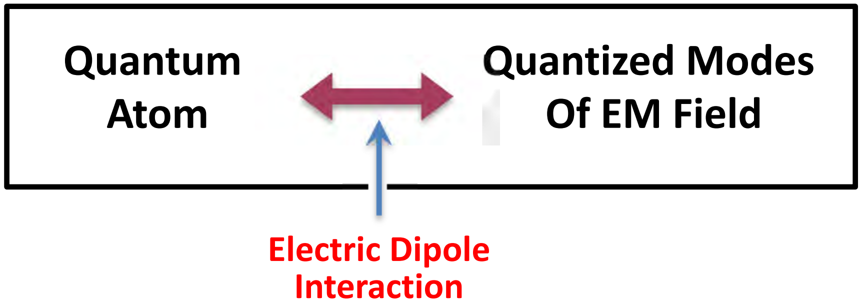
$\hat{E}_\perp,\ |\alpha(t)\rangle$ **or** $\hat{E}_\perp + E_\perp^{Cl}(\alpha, t),\ |0\rangle$

Validates Semiclassical Optics for strong Coherent Fields!

Quantized Light – Matter Interactions

Quantized Light – Matter Interactions

General Problem:



Quantum Atom

Quantized Modes Of EM Field

Electric Dipole Interaction

Starting Point: System Hamiltonian

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} \qquad \textbf{(1)}$$

$$\hat{H}_F = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \left(\hat{a}^{+}_{\vec{k}} \hat{a}_{\vec{k}} + 1/2\right) \qquad \textbf{Field}$$

$$\hat{H}_A = \sum_i E_i |i\rangle\langle i| = \sum_i E_i \hat{\sigma}_{ii} \qquad \textbf{Atom}$$

$$\hat{H}_{AF} = -\hat{\vec{p}} \cdot \hat{\vec{E}}(\vec{r},t) \qquad \textbf{ED interaction}$$

E_i, $|i\rangle$ **: energies, energy levels of the atom**