Quantum Electrodynamics – QED

Starting point: Maxwells Equations

(1) $\nabla \cdot \vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0}\rho(\vec{r},t)$

(2) $\nabla \cdot \vec{B}(\vec{r},t) = 0$

(3) $\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t)$

(4) $\nabla \times \vec{B}(\vec{r},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{r},t) + \frac{1}{\varepsilon_0 c^2}\vec{j}(\vec{r},t)$

Implicit: Charges & Fields in Vacuum
No "medium response"

Same issue as with our introductory example:

Maxwells eqs are non-local

We need to put the classical description in proper form -> Normal Mode expansion

Free Fields - Switch to Fourier Domain

(1) $i\vec{k} \cdot \vec{\mathcal{E}}(\vec{k},t) = \frac{1}{\varepsilon_0}\rho(\vec{k},t)$

(2) $i\vec{k} \cdot \vec{\mathcal{B}}(\vec{k},t) = 0$

(3) $i\vec{k} \times \vec{\mathcal{E}}(\vec{k},t) = -\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t)$

(4) $i\vec{k} \times \vec{\mathcal{B}}(\vec{k},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathcal{E}}(\vec{k},t) + \frac{1}{\varepsilon_0 c^2}\vec{\mathcal{J}}(\vec{k},t)$

Fourier Transform: $\begin{cases} \nabla \cdot \vec{G} \rightleftarrows i\vec{k} \cdot \vec{\mathcal{G}} \\ \nabla \times \vec{G} \rightleftarrows i\vec{k} \times \vec{\mathcal{G}} \end{cases}$

<u>Note</u>: This is a Normal Mode decomposition

No charges -> No coupling between modes with different $\vec{k}$

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Free Fields - Switch to Fourier Domain

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Separate into Transverse & Longitudinal Fields

$$\vec{\mathcal{E}}(\vec{k},t) = \vec{\mathcal{E}}_{\parallel}(\vec{k},t) + \vec{\mathcal{E}}_{\perp}(\vec{k},t)$$

$$\vec{\mathcal{B}}(\vec{k},t) = \cancel{\vec{\mathcal{B}}_{\parallel}(\vec{k},t)} + \vec{\mathcal{B}}_{\perp}(\vec{k},t) \quad \text{MEq (2)}$$

Entirely Transverse

<u>Note</u>:
$$\begin{cases} \vec{\mathcal{E}}_{\parallel} \text{ is } \frac{\vec{k}}{k} \times \text{ the projection of } \vec{\mathcal{E}} \text{ onto } \vec{k} \\ \mathcal{E}_{\parallel} = -\frac{i}{k}\, i\vec{k}\cdot\vec{\mathcal{E}} \text{ is the projection of } \vec{\mathcal{E}} \text{ onto } \vec{k} \end{cases}$$

$$\vec{\mathcal{E}}_{\parallel} = \frac{\vec{k}}{k}\mathcal{E}_{\parallel} = \frac{\vec{k}}{k}\left(-\frac{i}{k}\, i\vec{k}\cdot\vec{\mathcal{E}}\right) = \frac{\vec{k}}{\varepsilon_0 k^2}\rho(\vec{k},t)$$

Coulomb field from the charges

Only $\vec{\mathcal{E}}_{\perp}$ and $\vec{\mathcal{B}}_{\perp}$ are new degrees of freedom beyond the particles -> <u>Free Fields</u>

Quantum Electrodynamics – QED

Separate into Transverse & Longitudinal Fields

$$\vec{\mathcal{E}}(\vec{k},t) = \vec{\mathcal{E}}_{\parallel}(\vec{k},t) + \vec{\mathcal{E}}_{\perp}(\vec{k},t)$$

$$\vec{\mathcal{B}}(\vec{k},t) = \cancel{\vec{\mathcal{B}}_{\parallel}(\vec{k},t)} + \vec{\mathcal{B}}_{\perp}(\vec{k},t) \quad \text{MEq (2)}$$

Entirely Transverse

Note:
- $\vec{\mathcal{E}}_{\parallel}$ is $\frac{\vec{k}}{k} \times$ the projection of $\vec{\mathcal{E}}$ onto $\vec{k}$
- $\mathcal{E}_{\parallel} = -\frac{i}{k} i \vec{k} \cdot \vec{\mathcal{E}}$ is the projection of $\vec{\mathcal{E}}$ onto $\vec{k}$

MEq (1)

$$\vec{\mathcal{E}}_{\parallel} = \frac{\vec{k}}{k} \mathcal{E}_{\parallel} = \frac{\vec{k}}{k}\left(-\frac{i}{k}\, i\vec{k}\cdot\vec{\mathcal{E}}\right) = \frac{\vec{k}}{\varepsilon_0 k^2}\,\rho(\vec{k},t)$$

Coulomb field from the charges

Only $\vec{\mathcal{E}}_{\perp}$ and $\vec{\mathcal{B}}_{\perp}$ are new degrees of freedom beyond the particles -> <u>Free Fields</u>

Eqs for Transverse Fields, from MEqs (3) & (4)

(5a) $\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t) = -i\vec{k} \times \vec{\mathcal{E}}_{\perp}(\vec{k},t)$

(6a) $\frac{\partial}{\partial t}\vec{\mathcal{E}}_{\perp}(\vec{k},t) = c^2 i\vec{k} \times \vec{\mathcal{B}}(\vec{k},t) - \frac{1}{\varepsilon_0}\vec{j}_{\perp}(\vec{k},t)$

inverse FT

(5b) $\frac{\partial}{\partial t}\vec{B}(\vec{r},t) = -\nabla \times \vec{E}_{\perp}(\vec{r},t)$

(6b) $\frac{\partial}{\partial t}\vec{E}_{\perp}(\vec{r},t) = c^2 \nabla \times \vec{B}(\vec{r},t) - \frac{1}{\varepsilon_0}\vec{j}_{\perp}(\vec{r},t)$

combine (5b) & (6b)

Wave Equation for the Free Fields

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_{\perp}(\vec{r},t) = 0$$

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Eqs for Transverse Fields, from MEqs (3) & (4)

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Wave Equation for the Free Fields

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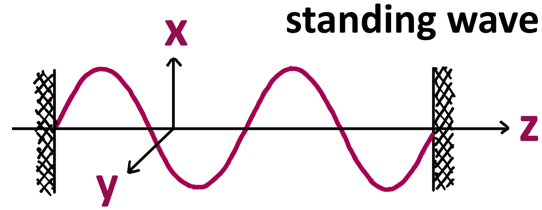
Normal Modes in a 1D Cavity

Length L

Cross section A

Volume $V = LA$



Normal Modes are Standing Waves

Let $\vec{E}(z,t) = \vec{\varepsilon}_x\, E_x(z,t)$ and expand

fiducial mass

(7) $$E_x(z,t) = \sum_j A_j\, q_j(t)\sin(k_j z),\quad A_j = \sqrt{\frac{2\omega_j m_j}{\varepsilon_0 V}}$$

MEq (4) w/no charges

$$\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}_\perp(\vec{r},t) = \vec{\varepsilon}_x\frac{1}{c^2}\sum_j A_j\,\dot{q}_j(t)\sin(k_j t)$$

$$= \vec{\varepsilon}_x\left(\cancel{\frac{\partial B_z}{\partial y}} - \frac{\partial B_y}{\partial z}\right) = -\vec{\varepsilon}_x\frac{\partial B_y}{\partial z}$$

$\mathcal{B}$ transverse ⇨ $B_z = 0$

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From Eq. (5a) we see that

$$\vec{B} \perp \vec{E}, \vec{\mathcal{E}}_z \Rightarrow \vec{B}(z,t) = \vec{\mathcal{E}}_y B_y(z,t)$$

Putting this together we get

$$\frac{\partial B_y}{\partial z} = -\sum_j \frac{A_j}{c^2} q_j(t) \sin(k_j z)$$

$$\boxed{(8) \quad B_y(z,t) = \sum_j \frac{A_j}{k_j c^2} \dot{q}_j(t) \cos(k_j z)}$$

Hamiltonian (Energy) for the Classical Field

$$\mathcal{H} = \frac{\varepsilon_0 A}{2} \int_0^L dz \left(|\vec{E}|^2 + c^2 |\vec{B}|^2\right) =$$

$$\frac{\varepsilon_0 A}{2} \int_0^L dz \sum_j \left[A_j^2 q_j(t)^2 \sin^2(k_j z) + \frac{A_j^2}{k_j} \dot{q}_j(t)^2 \cos^2(k_j z)\right]$$

V

Integrating over the Cavity volume

$$A \int_0^L dz \sin^2(k_j z) = A \int_0^L dz \cos^2(k_j z) = V/2$$

and substituting $A_j^2 = \frac{2\omega_j^2 m_j}{\varepsilon_0 V}$ we finally get

$$\boxed{\mathcal{H} = \sum_j \left[\frac{1}{2} m_j \omega_j^2 q_j^2 + \frac{1}{2} m_j \dot{q}_j^2\right]}$$

Lagrangian for the Classical Field

$$\boxed{\begin{aligned}\mathcal{L} &= \frac{\varepsilon_0 A}{2} \int_0^L dz \left(c^2 |\vec{B}|^2 - |\vec{E}|^2\right) \quad ☑ \\ &= \sum_j \left[\frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j \omega_j^2 q_j^2\right]\end{aligned}}$$

Check $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \vec{E}_\perp(\vec{r},t) = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

Quantum Electrodynamics – QED

Integrating over the Cavity volume

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Lagrangian for the Classical Field

$$\mathcal{L} = \frac{\varepsilon_0 A}{2}\int_0^L dz \left(c^2|\vec{B}|^2 - |\vec{E}|^2\right)$$

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$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_\perp(\vec{r}, t) = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

And Finally:

Conjugate Momentum $\boxed{p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = m_j \dot{q}_j}$

As before, a collection of Harmonic Oscillators, ready for quantization!

Quantum Electrodynamics – QED

Standard Quantization Procedure

$$q_j \to \hat{q}_j, \quad p_j \to \hat{p}_j, \qquad [\hat{q}_j, \hat{p}_{j'}] = i\hbar\,\delta_{jj'}$$

$$\alpha_j(t) \to \hat{a}_j, \quad \alpha_j^*(t) \to \hat{a}_j^\dagger, \qquad [\hat{a}_j, \hat{a}_{j'}^\dagger] = \delta_{jj'}$$

$$\hat{E}_x(z) = \sum_j \mathcal{E}_j\,(\hat{a}_j + \hat{a}_j^\dagger)\sin(k_j z)$$

$$\hat{B}_y(z) = -\frac{i}{c}\sum_j \mathcal{E}_j\,(\hat{a}_j - \hat{a}_j^\dagger)\cos(k_j z)$$

Total Field

$$\hat{\vec{E}}(z) = \vec{\varepsilon}_x \hat{E}_x(z) + \vec{\varepsilon}_y \hat{E}_y(z)$$

$$\hat{\vec{B}}(z) = \vec{\varepsilon}_x \hat{B}_x(z) + \vec{\varepsilon}_y \hat{B}_y(z)$$

Note:

These are the Field Operators in the Schrödinger Picture (*t*-dependence in states)

Often advantageous to use Heisenberg Picture (*t*-dependence in operators)

$$\alpha_j(t) \to \hat{a}_j(t) = \hat{a}_j(0)e^{-i\omega_j t}$$

Field Quantization in Free Space:

Normal Modes : $\vec{u}_{\vec{k},\lambda}(\vec{r}) = \vec{\mathcal{E}}_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k}} t - \vec{k}\cdot\vec{r})} + c.c.$

λ: polarization index

Finite quantization volume: $\mathcal{E}_{\vec{k}} = \sqrt{\hbar\omega_j / 2\varepsilon_0 V}$

$L \times L \times L$

L **large -> nature of boundary conditions not important**

Periodic boundary conditions

$$|\vec{k}| = n\,2\pi/L$$

Quantum Electrodynamics – QED

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Classical Fields (Fourier Sum):

$$\vec{E}_\perp(\vec{r},t) = \sum_{\vec{k},\lambda} \vec{\varepsilon}_{\vec{k},\lambda}\, \mathcal{E}_{\vec{k},\lambda}\, \alpha_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k}} t - \vec{k}\cdot\vec{r})} + c.c.$$

$$\vec{B}_\perp(\vec{r},t) = \sum_{\vec{k},\lambda} \frac{\vec{k}\times\vec{\varepsilon}_{\vec{k},\lambda}}{kc}\, \mathcal{E}_{\vec{k},\lambda}\, \alpha_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k}} t - \vec{k}\cdot\vec{r})} + c.c.$$

Quantization:

$$\alpha_{\vec{k},\lambda} \rightarrow \hat{a}_{\vec{k},\lambda}, \quad [\hat{a}_{\vec{k},\lambda}, \hat{a}^{+}_{\vec{k}',\lambda'}] = \delta_{\vec{k},\vec{k}'}\delta_{\lambda,\lambda'}$$

$$\alpha^{*}_{\vec{k},\lambda} \rightarrow \hat{a}^{+}_{\vec{k},\lambda}, \quad [\hat{a}_{\vec{k},\lambda}, \hat{a}_{\vec{k}',\lambda'}] = [\hat{a}^{+}_{\vec{k},\lambda}, \hat{a}^{+}_{\vec{k}',\lambda'}] = 0$$

$$\hat{\vec{E}}_\perp(\vec{r},t) = \sum_{\vec{k},\lambda} \vec{\varepsilon}_{\vec{k},\lambda}\, \mathcal{E}_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k},\lambda} t - \vec{k}\cdot\vec{r})} + H.C.$$

$$\hat{\vec{B}}(\vec{r},t) = \sum_{\vec{k},\lambda} \frac{\vec{k}\times\vec{\varepsilon}_{\vec{k},\lambda}}{kc}\, \mathcal{E}_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k},\lambda} t - \vec{k}\cdot\vec{r})} + H.C.$$

– Heisenberg Picture –

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Positive & Negative Frequency Components:

$$\hat{\vec{E}}_\perp(\vec{r},t) = \hat{\vec{E}}^{(+)}(\vec{r},t) + \hat{\vec{E}}^{(-)}(\vec{r},t)$$

$$\hat{\vec{E}}^{(+)}(\vec{r},t) = \sum_{\vec{k},\lambda} i\vec{\varepsilon}_{\vec{k},\lambda}\,\mathcal{E}_{\vec{k},\lambda}\,\hat{a}_{\vec{k},\lambda}\,e^{i(\vec{k}\cdot\vec{r}-\omega_{\vec{k},\lambda}t)}$$

$$\hat{\vec{E}}^{(-)}(\vec{r},t) = \sum_{\vec{k},\lambda} -i\vec{\varepsilon}^{\,*}_{\vec{k},\lambda}\,\mathcal{E}^{*}_{\vec{k},\lambda}\,\hat{a}^{+}_{\vec{k},\lambda}\,e^{-i(\vec{k}\cdot\vec{r}-\omega_{\vec{k},\lambda}t)}$$

Wrap Up:

Read page 13 in handwritten Note Set for brief discussion of different, equivalent ways to put the QED formalism together, e. g.

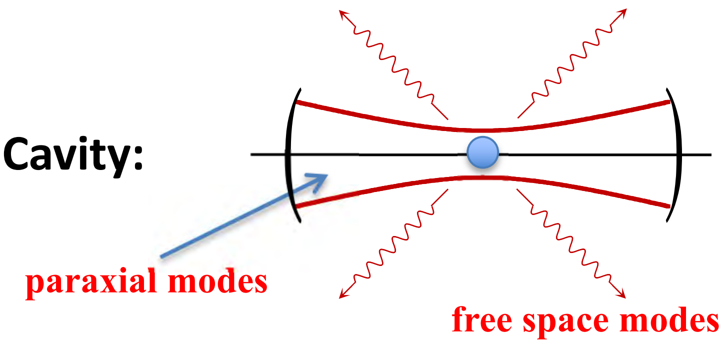
$$\hat{E}_x \propto (\hat{a}_j + \hat{a}_j^{+}) \quad \& \quad \hat{B}_y \propto (\hat{a}_j - \hat{a}_j^{+})$$

vs

$$\hat{E}_x \propto (\hat{a}_j - \hat{a}_j^{+}) \quad \& \quad \hat{B}_y \propto (\hat{a}_j + \hat{a}_j^{+})$$

Other Normal Modes Sets

Atom in Cavity:



Wavepackets: (Milloni & Eberly, Sec. 12.8, p 381) (QED lecture notes, p 16)

Classical field pulse envelope

$$\vec{E}(\vec{r},t) = \vec{\varepsilon}\,\mathcal{E}_0\,u(z-ct)\,e^{i(k_0 z-\omega_0 t)} + c.c.$$

Mode volume $V = \int d^3r\,|u(x,y,z-ct)|^2$

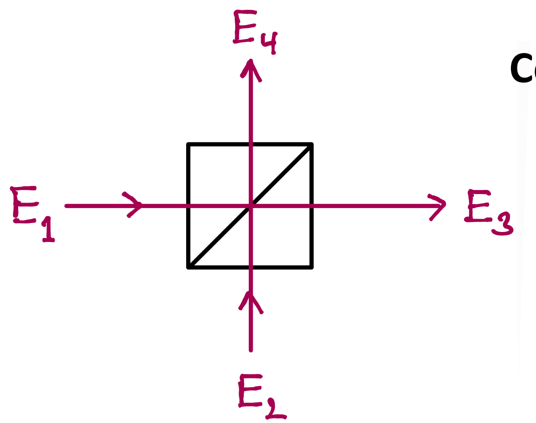
Quantization $\mathcal{E}_0 \to \mathcal{E}_k \alpha_k \to \mathcal{E}_k \hat{a}_k$ **etc.**

Wave-Particle Duality similar for Photons and Phonons

Application: Classical & Quantum Beamsplitters

Application: Classical & Quantum Beamsplitters

Classical Beamsplitter



Coupled H & V modes

Linear symmetric input-output map

$$E_3 = tE_1 + rE_2$$
$$E_4 = rE_1 + tE_2$$

Energy conservation requires

$$|E_1|^2 + |E_2|^2 = |E_3|^2 + |E_4|^2$$

Choose $E_1 = E \ , \ E_2 = 0$ ⇨

$$|E_3|^2 + |E_4|^2 = E_0\left(|t|^2 + |r|^2\right)$$

Choose $E_1 = \frac{1}{\sqrt{2}} E \ , \ E_2 = \frac{1}{\sqrt{2}} E_0$ ⇨

$$|E_3|^2 + |E_4|^2 = \frac{1}{2} E_0 |t + r|^2$$ ⇨

$$|t|^2 + |r|^2 + tr^* + rt^* = 1$$

From this it follows that

$$\boxed{\begin{aligned} |t|^2 + |r|^2 &= 1 \\ tr^* + rt^* &= 0 \end{aligned}}$$

Classical input-output map

$$\boxed{\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}}$$

Quantum Beamsplitter

Heisenberg Picture ⇨ **Field Operators obey Maxwells Eqs**

Classical field	Quantum equivalent
$E_\perp(\vec{r},t) \propto \alpha(t)$	$\hat{E}_\perp^{(+)}(\vec{r},t) \propto \hat{a}(t)$

Application: Classical & Quantum Beamsplitters

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Quantum Beamsplitter

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$$E_\perp(\vec{r},t)\propto\alpha(t)$$

Quantum equivalent

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Quantum Beamsplitter

$$\begin{pmatrix}\hat{E}_3\\\hat{E}_4\end{pmatrix}=\begin{pmatrix}t & r\\ r & t\end{pmatrix}\begin{pmatrix}\hat{E}_1\\\hat{E}_2\end{pmatrix}$$

Quantum input-output map

$$\boxed{\begin{pmatrix}\hat{a}_3\\\hat{a}_4\end{pmatrix}=\begin{pmatrix}t & r\\ r & t\end{pmatrix}\begin{pmatrix}\hat{a}_1\\\hat{a}_2\end{pmatrix}}$$

Invert Map

$$\begin{aligned}\hat{a}_3&=t\hat{a}_1+r\hat{a}_2\\ \hat{a}_4&=r\hat{a}_1+t\hat{a}_2\end{aligned}\quad\Rightarrow\quad\begin{aligned}\hat{a}_1&=t^*\hat{a}_3+r^*\hat{a}_4\\ \hat{a}_2&=r^*\hat{a}_3+t^*\hat{a}_4\end{aligned}$$

Switch to creation operators ⇒

$$\boxed{\begin{aligned}\hat{a}_1^+&=t\hat{a}_3^++r\hat{a}_4^+\\ \hat{a}_2^+&=r\hat{a}_3^++t\hat{a}_4^+\end{aligned}}$$

Application: Classical & Quantum Beamsplitters

Quantum Beamsplitter

$$\begin{pmatrix} \hat{E}_3 \\ \hat{E}_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \hat{E}_1 \\ \hat{E}_2 \end{pmatrix}$$

Quantum input-output map

$$\boxed{\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}}$$

Invert Map

$$\hat{a}_3 = t\hat{a}_1 + r\hat{a}_2 \qquad \hat{a}_1 = t^*\hat{a}_3 + r^*\hat{a}_4$$

$$\hat{a}_4 = r\hat{a}_1 + t\hat{a}_2 \qquad \hat{a}_2 = r^*\hat{a}_3 + t^*\hat{a}_4$$

Switch to creation operators

$$\boxed{\begin{aligned} \hat{a}_1^\dagger &= t\hat{a}_3^\dagger + r\hat{a}_4^\dagger \\ \hat{a}_2^\dagger &= r\hat{a}_3^\dagger + t\hat{a}_4^\dagger \end{aligned}}$$

Switch to Schrödinger Picture

General input state: 2-mode vacuum

$$\boxed{|\psi_{in}\rangle = \sum_{nm} f_n \frac{1}{\sqrt{n!}} (\hat{a}_1^\dagger)^n g_m \frac{1}{\sqrt{m!}} (\hat{a}_2^\dagger)^m |0\rangle}$$

The BS maps $\hat{a}_1^\dagger, \hat{a}_2^\dagger$ to linear combinations of $\hat{a}_3^\dagger, \hat{a}_4^\dagger$

General output state: (Schrödinger Picture)

$$\boxed{|\psi_{out}\rangle = \sum_{nm} f_n \frac{1}{\sqrt{n!}} (t\hat{a}_3^\dagger + r\hat{a}_4^\dagger)^n g_m \frac{1}{\sqrt{m!}} (r\hat{a}_3 + t\hat{a}_4)^m |0\rangle}$$

Example: One-photon input state

$$|\psi_{in}\rangle = |1\rangle_1 |0\rangle_2 = \hat{a}_1^\dagger |0\rangle$$

$$|\psi_{out}\rangle = (t\hat{a}_3^\dagger + r\hat{a}_4^\dagger)|0\rangle = t|1\rangle_3|0\rangle_4 + r|0\rangle_3|1\rangle_4$$

Application: Classical & Quantum Beamsplitters

Switch to Schrödinger Picture

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General output state: (Schrödinger Picture)

$$\boxed{|\psi_{out}\rangle = \sum_{nm} f_n \frac{1}{\sqrt{n!}}(t\hat{a}_3^{\dagger} + r\hat{a}_4^{\dagger})^n g_m \frac{1}{\sqrt{m!}}(r\hat{a}_3 + t\hat{a}_4)^m |0\rangle}$$

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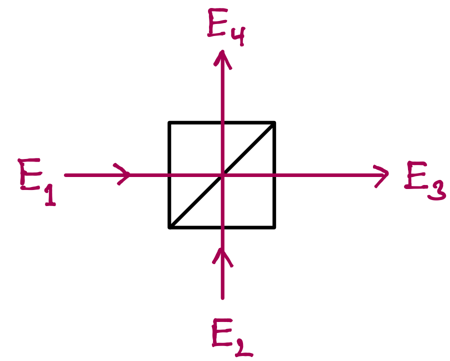
50/50 Beamsplitter



$$t = 1/\sqrt{2}, \quad r = i/\sqrt{2}$$

$$\boxed{|\psi_{out}\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle_3|0\rangle_4 + i|0\rangle_3|1\rangle_4\right)}$$

Note: This is a **Mode Entangled State**

(*) A coherent superposition of states w/ one photon in port 3 and zero in port 4, and zero in port 3 and one in port 4.

Can we assign states such as, e. g.

$\frac{1}{\sqrt{2}}\left(|1\rangle_3 + i|0\rangle_3\right)$ to port 3

?

$\frac{1}{\sqrt{2}}\left(|0\rangle_4 + i|1\rangle_4\right)$ to port 4

Viewed on their own, each port is in a <u>mixed state</u>

Application: Classical & Quantum Beamsplitters

Switch to Schrödinger Picture

General input state: 2-mode vacuum

$$\boxed{|\psi_{in}\rangle = \sum_{nm} f_n \frac{1}{\sqrt{n!}}(\hat{a}_1^\dagger)^n g_m \frac{1}{\sqrt{m!}}(\hat{a}_2^\dagger)^m |0\rangle}$$

The BS maps $\hat{a}_1^\dagger, \hat{a}_2^\dagger$ to linear combinations of $\hat{a}_3^\dagger, \hat{a}_4^\dagger$

General output state: (Schrödinger Picture)

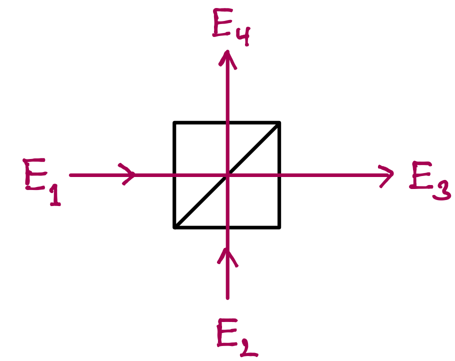
$$\boxed{|\psi_{out}\rangle = \sum_{nm} f_n \frac{1}{\sqrt{n!}}(t\hat{a}_3^\dagger + r\hat{a}_4^\dagger)^n g_m \frac{1}{\sqrt{m!}}(r\hat{a}_3 + t\hat{a}_4)^m |0\rangle}$$

Example: One-photon input state

$$|\psi_{in}\rangle = |1\rangle_1|0\rangle_2 = \hat{a}_1^\dagger|0\rangle$$

$$|\psi_{out}\rangle = (t\hat{a}_3^\dagger + r\hat{a}_4^\dagger)|0\rangle = t|1\rangle_3|0\rangle_4 + r|0\rangle_3|1\rangle_4$$

50/50 Beamsplitter



$$t = 1/\sqrt{2},\ r = i/\sqrt{2}$$

$$\boxed{|\psi_{out}\rangle = \frac{1}{\sqrt{2}}(|1\rangle_3|0\rangle_4 + i|0\rangle_3|1\rangle_4)}$$

Note: This is a **Mode Entangled State**

(*) A coherent superposition of states w/ one photon in port 3 and zero in port 4, and zero in port 3 and one in port 4.

Can we assign states such as, e. g.

$\cancel{\frac{1}{\sqrt{2}}(|1\rangle_3 + i|0\rangle_3)}$ to port 3

$\cancel{\frac{1}{\sqrt{2}}(|0\rangle_4 + i|1\rangle_4)}$ to port 4

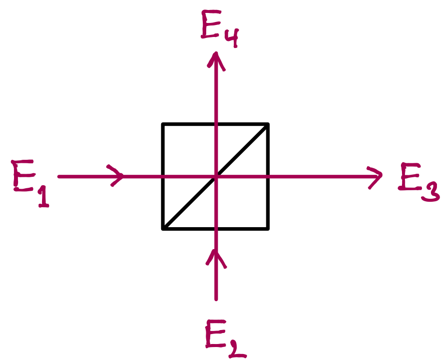
!

Viewed on their own, each port is in a <u>mixed state</u>

Application: Classical & Quantum Beamsplitters

50/50 Beamsplitter

$t = 1/\sqrt{2}, \; r = i/\sqrt{2}$



$$\boxed{|\psi_{out}\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle_3|0\rangle_4 + i|0\rangle_3|1\rangle_4\right)}$$

Note: This is a Mode Entangled State

(*) A coherent superposition of states w/ one photon in port 3 and zero in port 4, and zero in port 3 and one in port 4.

Can we assign states such as, e. g.

$\cancel{\frac{1}{\sqrt{2}}\left(|1\rangle_3 + i|0\rangle_3\right)}$ to port 3

$\cancel{\frac{1}{\sqrt{2}}\left(|0\rangle_4 + i|1\rangle_4\right)}$ to port 4

!

Viewed on their own, each port is in a <u>mixed state</u>

Example: Two-photon input state, 50/50 BS

$$|\psi_{in}\rangle = \hat{a}_1^\dagger \hat{a}_2^\dagger |0\rangle$$

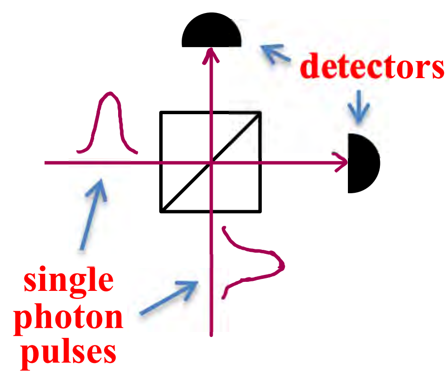
$$|\psi_{out}\rangle = \frac{1}{2}\left(\hat{a}_3^\dagger + i\hat{a}_4^\dagger\right)\left(i\hat{a}_3^\dagger + \hat{a}_4^\dagger\right)|0\rangle$$

destructive interference

$$= \frac{1}{2}\left(i\hat{a}_3^\dagger\hat{a}_3^\dagger + i\hat{a}_4^\dagger\hat{a}_4^\dagger + \hat{a}_3^\dagger\hat{a}_4^\dagger - \hat{a}_4^\dagger\hat{a}_3^\dagger\right)|0\rangle$$

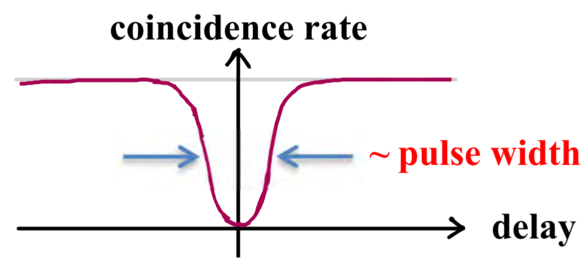
$$= \frac{i}{2}\left(\hat{a}_3^\dagger\hat{a}_3^\dagger + \hat{a}_4^\dagger\hat{a}_4^\dagger\right)|0\rangle = \frac{i}{\sqrt{2}}\left(|2\rangle_3|0\rangle_4 + |0\rangle_3|2\rangle_4\right)$$

Experiment:



Coincidence detections are never seen when pulses overlap -> "bunching".

Delay between pulses leads to Coincidence detections.



Application: Classical & Quantum Beamsplitters

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Measurement of Subpicosecond Time Intervals between Two Photons by Interference

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A fourth-order interference technique has been used to measure the time intervals between two photons, and by implication the length of the photon wave packet, produced in the process of parametric down-conversion. The width of the time-interval distribution, which is largely determined by an interference filter, is found to be about 100 fs, with an accuracy that could, in principle, be less than 1 fs.

PACS numbers: 42.50.Bs, 42.65.Re

The usual way to determine the duration of a short pulse of light is to superpose two similar pulses and to measure the overlap with a device having a nonlinear response.[1] The latter might, for example, make use of the process of harmonic generation in a nonlinear medium. Indeed, such a technique was recently used[2] to determine the coherence length of the light generated in the process of parametric down-conversion.[3] The coherence time was found to be of subpicosecond duration, as predicted theoretically.[4] It is, however, in the nature of the technique that it requires very intense light pulses and would be of no use for the measurement of single

phasized that the signal and idler photons have no definite phase, and are therefore mutually incoherent, in the sense that they exhibit no second-order interference when brought together at detector D1 or D2. However, fourth-order interference effects occur, as demonstrated by the coincidence counting rate between D1 and D2.[6–8] The experiment has some similarities to another, recently reported, two-photon interference experiment in which fringes were observed and measured, but without the use of a beam splitter.[6]

Although the sum frequency $\omega_1+\omega_2$ is very well defined in the experiment, the individual down-shifted

Application: Classical & Quantum Beamsplitters

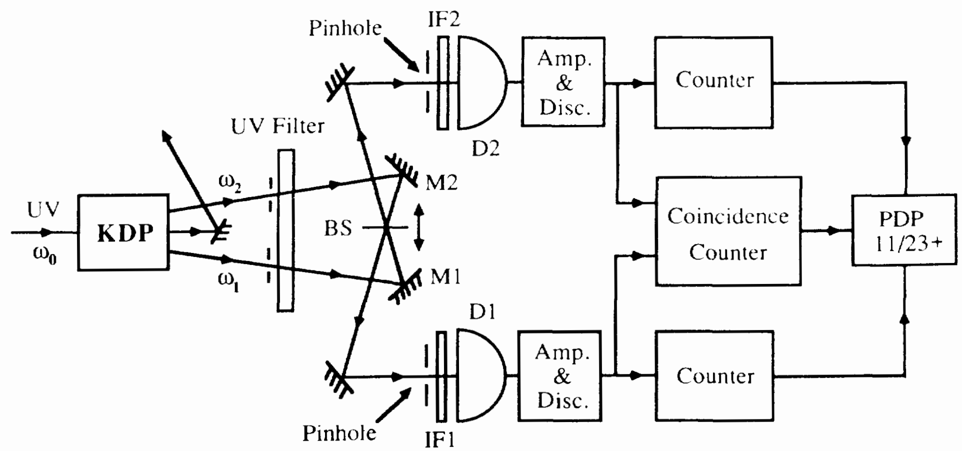

FIG. 1. Outline of the experimental setup.