Quantum Electrodynamics – QED

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Starting point: Maxwells Equations

(1) $\nabla \cdot \vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0}\rho(\vec{r},t)$

(2) $\nabla \cdot \vec{B}(\vec{r},t) = 0$

(3) $\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t)$

(4) $\nabla \times \vec{B}(\vec{r},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{r},t) + \frac{1}{\varepsilon_0 c^2}\vec{j}(\vec{r},t)$

Implicit: **Charges & Fields in Vacuum**
No "medium response"

Same issue as with our introductory example:

Maxwells eqs are non-local

We need to put the classical description in proper form -> Normal Mode expansion

Free Fields - Switch to Fourier Domain

(1) $i\vec{k} \cdot \vec{\mathcal{E}}(\vec{k},t) = \frac{1}{\varepsilon_0}\rho(\vec{k},t)$

(2) $i\vec{k} \cdot \vec{\mathcal{B}}(\vec{k},t) = 0$

(3) $i\vec{k} \times \vec{\mathcal{E}}(\vec{k},t) = -\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t)$

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Fourier Transform:
$$\left\{ \begin{array}{l} \nabla \cdot \vec{G} \rightleftarrows i\vec{k} \cdot \vec{\mathcal{G}} \\ \nabla \times \vec{G} \rightleftarrows i\vec{k} \times \vec{\mathcal{G}} \end{array} \right.$$

<u>Note</u>:   This is a Normal Mode decomposition

No charges -> No coupling between modes with different $\vec{k}$

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Separate into Transverse & Longitudinal Fields

$$\vec{\mathcal{E}}(\vec{k},t) = \vec{\mathcal{E}}_{\parallel}(\vec{k},t) + \vec{\mathcal{E}}_{\perp}(\vec{k},t)$$

$$\vec{\mathcal{B}}(\vec{k},t) = \cancel{\vec{\mathcal{B}}_{\parallel}(\vec{k},t)} + \vec{\mathcal{B}}_{\perp}(\vec{k},t) \quad \text{MEq (2)}$$

Entirely Transverse

Note:
$$\begin{cases} \vec{\mathcal{E}}_{\parallel} \text{ is } \frac{\vec{k}}{k} \times \text{ the projection of } \vec{\mathcal{E}} \text{ onto } \vec{k} \\ \mathcal{E}_{\parallel} = -\frac{i}{k}\, i\vec{k}\cdot\vec{\mathcal{E}} \text{ is the projection of } \vec{\mathcal{E}} \text{ onto } \vec{k} \end{cases}$$

$$\vec{\mathcal{E}}_{\parallel} = \frac{\vec{k}}{k}\mathcal{E}_{\parallel} = \frac{\vec{k}}{k}\left(-\frac{i}{k}\, i\vec{k}\cdot\vec{\mathcal{E}}\right) = \frac{\vec{k}}{\varepsilon_0 k^2}\rho(\vec{k},t)$$

Coulomb field from the charges

Only $\vec{\mathcal{E}}_{\perp}$ and $\vec{\mathcal{B}}_{\perp}$ are new degrees of freedom beyond the particles -> **Free Fields**

Quantum Electrodynamics – QED

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Note:
- $\vec{\mathcal{E}}_{\parallel}$ is $\frac{\vec{k}}{k} \times$ the projection of $\vec{\mathcal{E}}$ onto $\vec{k}$
- $\mathcal{E}_{\parallel} = -\frac{i}{k} i\vec{k}\cdot\vec{\mathcal{E}}$ is the projection of $\vec{\mathcal{E}}$ onto $\vec{k}$

MEq (1)

$$\vec{\mathcal{E}}_{\parallel} = \frac{\vec{k}}{k}\mathcal{E}_{\parallel} = \frac{\vec{k}}{k}\left(-\frac{i}{k} i\vec{k}\cdot\vec{\mathcal{E}}\right) = \frac{\vec{k}}{\varepsilon_0 k^2}\rho(\vec{k},t)$$

Coulomb field from the charges

Only $\vec{\mathcal{E}}_{\perp}$ and $\vec{\mathcal{B}}_{\perp}$ are new degrees of freedom beyond the particles -> <u>Free Fields</u>

Eqs for Transverse Fields, from MEqs (3) & (4)

(5a) $$\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t) = -i\vec{k}\times\vec{\mathcal{E}}_{\perp}(\vec{k},t)$$

(6a) $$\frac{\partial}{\partial t}\vec{\mathcal{E}}_{\perp}(\vec{k},t) = c^2 i\vec{k}\times\vec{\mathcal{B}}(\vec{k},t) - \frac{1}{\varepsilon_0}\vec{j}_{\perp}(\vec{k},t)$$

inverse FT

(5b) $$\frac{\partial}{\partial t}\vec{B}(\vec{r},t) = -\nabla\times\vec{E}_{\perp}(\vec{r},t)$$

(6b) $$\frac{\partial}{\partial t}\vec{E}_{\perp}(\vec{r},t) = c^2\nabla\times\vec{B}(\vec{r},t) - \frac{1}{\varepsilon_0}\vec{j}_{\perp}(\vec{r},t)$$

combine (5b) & (6b)

Wave Equation for the Free Fields

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_{\perp}(\vec{r},t) = \frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t}\vec{j}_{\perp}(\vec{r},t)$$

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Wave Equation for the Free Fields

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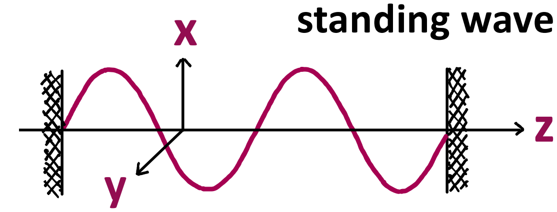
Normal Modes in a 1D Cavity

Length L

Cross section A

Volume $V = LA$



Normal Modes are Standing Waves

Let $\vec{E}(z,t) = \vec{\mathcal{E}}_x E_x(z,t)$ and expand

fiducial mass

(7) $E_x(z,t) = \sum_j A_j q_j(t)\sin(k_j z)\,,\quad A_j = \sqrt{\frac{2\omega_j m_j}{\varepsilon_0 V}}$

MEq (4) w/no charges

$$\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}_\perp(\vec{r},t) = \vec{\mathcal{E}}_x\frac{1}{c^2}\sum_j A_j\dot{q}_j(t)\sin(k_j t)$$

$$= \vec{\mathcal{E}}_x\left(\cancel{\frac{\partial B_z}{\partial y}} - \frac{\partial B_y}{\partial z}\right)$$

$\mathcal{B}$ transverse ⇨ $B_z = 0$

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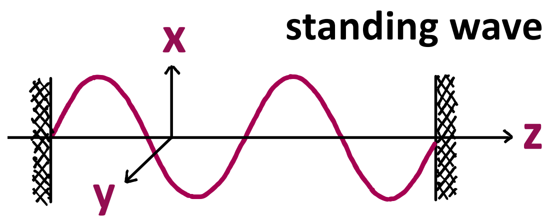

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$$\boxed{(7)\quad E_x(z,t) = \sum_j A_j q_j(t) \sin(k_j z), \quad A_j = \sqrt{\frac{2\omega_j m_j}{\varepsilon_0 V}}}$$

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$$= \vec{\varepsilon}_x \left(\cancel{\frac{\partial B_z}{\partial y}} - \frac{\partial B_y}{\partial z} \right) = -\vec{\varepsilon}_x \boxed{\frac{\partial B_y}{\partial z}}$$

$\mathcal{B}$ transverse $\Rightarrow$ $B_z = 0$

From Eq. (5a) we see that

$$\vec{B} \perp \vec{E}, \vec{\varepsilon}_z \Rightarrow \vec{B}(z,t) = \vec{\varepsilon}_y B_y(z,t)$$

Putting this together we get

$$\frac{\partial B_y}{\partial z} = -\sum_j \frac{A_j}{c^2} q_j(t) \sin(k_j z)$$

$$\boxed{(8)\quad B_y(z,t) = \sum_j \frac{A_j}{k_j c^2} \dot{q}_j(t) \cos(k_j z)}$$

Hamiltonian (Energy) for the Classical Field

$$\mathcal{H} = \frac{\varepsilon_0 A}{2} \int_0^L dz \left(|\vec{E}|^2 + c^2 |\vec{B}|^2 \right) =$$

$$\frac{\varepsilon_0 A}{2} \int_0^L dz \sum_j \left[A_j^2 q_j(t)^2 \sin^2(k_j z) + \frac{A_j^2}{k_j} \dot{q}_j(t)^2 \cos^2(k_j z) \right]$$

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$$\Downarrow$$

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Integrating over the Cavity volume

$$A \int_0^L dz \sin^2(k_j z) = A \int_0^L dz \cos^2(k_j z) = V/2$$

and substituting $A_j^2 = \frac{2\omega_j^2 m_j}{\varepsilon_0 V}$ we finally get

$$\boxed{\mathcal{H} = \sum_j \left[\frac{1}{2} m_j \omega_j^2 q_j^2 + \frac{1}{2} m_j \dot{q}_j^2 \right]}$$

Lagrangian for the Classical Field

$$\boxed{\begin{aligned} \mathcal{L} &= \frac{\varepsilon_0 A}{2} \int_0^L dz \left(c^2 |\vec{B}|^2 - |\vec{E}|^2 \right) \\ &= \sum_j \left[\frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j \omega_j^2 q_j^2 \right] \end{aligned}}$$

Check $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E}_\perp(\vec{r}, t) = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$$

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And Finally:

Conjugate Momentum $\boxed{p_j = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_j} = m_j \dot{q}_j}$

As before, a collection of Harmonic Oscillators, ready for quantization!