# OPTI 544 Solution Set 6, Spring 2025 (Updated)

**Electronic Submission Only, by email to Jon Pajaud ( jpajaud@email.arizona.edu )**

## Problem 1

**Note:** It is non-trivial to obtain the wave equation from the Lagrangian expressed in terms of the acoustic field $\eta(x)$, as this involves taking the derivative of a functional $(\mathcal{L})$ with respect to functions $(\eta(x),\dot{\eta}(x))$. We avoid the need to learn about functional derivatives by stating from the discrete Lagrangian,

$$\mathcal{L} = \frac{1}{2}\sum_i m\dot{x}_i^2 - \kappa\left(x_{i+1}-x_i\right)^2 = \frac{1}{2}\sum_i a\left[\frac{m}{a}\dot{x}_i^2 - \kappa a\left(\frac{x_{i+1}-x_i}{a}\right)^2\right]$$

From the Lagrange equation of motion, $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_i} - \frac{\partial\mathcal{L}}{\partial x_i} = 0$, we get an equation for each *i*:

$$\frac{m}{a}\ddot{x} - \frac{\kappa a}{a}\left(\frac{x_{i+1}-x_i}{a}\right) + \frac{\kappa a}{a}\left(\frac{x_i - x_{i-1}}{a}\right) = 0 \qquad \text{Eq. (i)}$$

In the limit $a \to 0$ the last two terms become

$$\lim_{a\to 0}\left\{-\frac{Y}{a}\left[\left(\frac{\partial\eta}{\partial x}\right)_{x_i} - \left(\frac{\partial\eta}{\partial x}\right)_{x_i - a}\right]\right\} = -Y\frac{\partial^2\eta}{\partial x^2}$$

Also $$\lim_{a\to 0}\left\{\frac{m}{a}\ddot{x}_i\right\} = \mu\frac{\partial^2\eta}{\partial t^2}$$

Thus Eq. (i) above turns into a wave equation for the acoustic field,

$$\boxed{\mu\frac{\partial^2\eta}{\partial t^2} - Y\frac{\partial^2\eta}{\partial x^2} = 0}$$

**Problem 2**

(a) Expressed in terms of the field $\eta(x)$, the kinetic energy is

$$T = \int dx \frac{1}{2}\mu\left(\frac{\eta(x,t)}{dt}\right)^2 = \sum_{k,k'} \frac{1}{2}\mu L\, \dot{q}_k \dot{q}_{k'} \int dx\, u_k(x)\, u_{k'}(x)$$
$$= \sum_{k,k'} \frac{1}{2} M \dot{q}_k^2$$

(b) Working out the expression for the potential energy is a bit more involved. First

$$V = \int dx \frac{1}{2} Y \left(\frac{\eta(x,t)}{dx}\right)^2 = \sum_{k,k'} \frac{1}{2} LY\, q_k q_{k'} \int dx\, \tfrac{d}{dx} u_k(x) \tfrac{d}{dx} u_{k'}(x) \tag{1}$$

Using integration by part, $\int f(x) G(x)\, dx = F(x) G(x) - \int F(x) g(x)\, dx$, we can rewrite the last part,

$$\int dx \frac{d}{dx} u_k(x) \frac{d}{dx}\, u_{k'}(x) = u_k(x) \frac{d}{dx} u_{k'}(x) - \int dx\, u_k(x) \frac{d^2}{dx^2}\, u_{k'}(x)$$

Substituting, we get

$$V = \sum_{k,k'} \frac{1}{2} LY\, q_k q_{k'} \left[ u_k(x) \frac{d}{dx} u_{k'}(x) - \int dx\, u_k(x) \frac{d^2}{dx^2}\, u_{k'}(x) \right] \tag{2}$$

Next, we use

$$\sum_{k,k'} u_k(x) \frac{d}{dx} u_{k'}(x) = \frac{1}{2} \frac{d}{dx} \sum_{k,k'} u_k(x) u_{k'}(x) = \frac{1}{2} \frac{d^2}{dx^2} \sum_{k,k'} \int dx\, u_k(x) u_{k'}(x) = \frac{1}{2} \frac{d^2}{dx^2} \sum_{k,k'} \delta_{k,k'} = 0$$

where in the last step we have used $\int dx\, u_k(x) u_{k'}(x) = \delta_{kk'}$. Substituting in (2) we get

$$V = \sum_{k,k'} \frac{1}{2} LY q_k q_{k'} \left[ \int dx \left( -\frac{d^2}{dx^2} u_k(x) \right) u_{k'}(x) \right] \tag{3}$$

Finally, we use

$$\frac{d^2}{dx^2}u_k(x) = -k^2 u_k(x) \Rightarrow \int dx\, \frac{d^2}{dx^2}u_k(x)u_{k'}(x) = k^2 \int dx u_k(x) u_{k'}(x) = k^2 \delta_{kk'} ,$$

where $k^2 = \frac{\omega_k^2}{v^2} = \frac{\mu}{Y}\omega_k^2$. Substituting in (3), we then get

$$V = \sum_{k,k'} \frac{1}{2} LY q_k q_{k'} k^2 \delta_{kk'} = \sum_k \frac{1}{2} LY \frac{\mu}{Y} \omega_k^2 q_k^2 = \sum_k \frac{1}{2} M\omega^2 q_k^2 \qquad (4)$$

This is the result given in the notes.

(c) The Lagrangian is

$$\mathcal{L} = T - V = \int dx \frac{1}{2}\mu\left(\frac{\eta(x,t)}{dt}\right)^2 - \int dx \frac{1}{2} Y\left(\frac{\eta(x,t)}{dx}\right)^2 = \sum_k \frac{1}{2} M\dot{q}_k^2 - \sum_k \frac{1}{2} M\omega^2 q_k^2$$

Plugging into the Lagrange equation of motion gives us

$$\frac{d}{dt}\frac{\partial \mathcal{L}_k}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}_k}{\partial q_k} = M\ddot{q}_k - M\omega^2 q_k^2 \; = \; 0 \; \Rightarrow \; \ddot{q}_k + \omega^2 q_k^2 = 0$$

This is the standard differential equation for a collection of harmonic oscillators, one for each normal mode $k$.