Maxwell-Bloch Equations

So far in the Semiclassical Description

(*) Classical light acting on quantum atoms

(*) Next: Close the loop

Self-Consistent Description

Electric Field → 2-Level Atom

We need to set up and solve a set of workable simultaneous equations for the atoms and field.

(1) The <u>electric field</u>.   We write

$$\boxed{\vec{E}(z,t) = \vec{\varepsilon}\,\mathcal{E}(z,t)\,e^{-i(\omega t - kz)}}$$

wavepacket envelope

– Plane wave propagating in the $+z$ direction, the real part is the physical field

Slowly Varying Envelope Approximation (SVEA)

We require that the envelope $\mathcal{E}(z,t)$ is smooth in space and time compared to the plane wave part.

$$\left|\frac{\partial \mathcal{E}}{\partial z}\right| \ll k|\mathcal{E}| \qquad \left|\frac{\partial \mathcal{E}}{\partial t}\right| \ll \omega|\mathcal{E}|$$

$$\left|\frac{\partial^2 \mathcal{E}}{\partial z^2}\right| \ll k\left|\frac{\partial \mathcal{E}}{\partial z}\right| \qquad \left|\frac{\partial^2 \mathcal{E}}{\partial z^2}\right| \ll \omega\left|\frac{\partial \mathcal{E}}{\partial z}\right|$$

This is not particularly restrictive, unless working with ultrafast lasers.

(2) <u>The Macroscopic Polarization Density</u>.

We use the quantum expectation value $\vec{P}(z,t) = N\langle\hat{\vec{\mu}}\rangle$

Of this, we need the complex part that goes with $\vec{E}(z,t)$ and can be plugged into the wave equation.

Maxwell-Bloch Equations

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Thus, of

$$\langle\vec{\mu}\rangle = \vec{\mu}_{12}\langle a_2 a_1^*\rangle + \vec{\mu}_{21}\langle a_1 a_2^*\rangle$$
$$= \vec{\mu}_{12}\,\rho_{21}\,e^{-i(\omega t - kz)} + \vec{\mu}_{21}\rho_{12}e^{i(\omega t - kz)}$$

slow variables

we need the part that goes as $e^{-i(\omega t - kz)}$

The *physical field* is $\mathrm{Re}[\vec{\varepsilon}\,\mathcal{E}(z,t)e^{-i(\omega t - kz)}]$

The *physical dipole* is

$$\mathrm{Re}\left[\boxed{\vec{\mu}_{12}\rho_{21}e^{-i(\omega t - kz)}} + \vec{\mu}_{21}\rho_{12}e^{i(\omega t - kz)}\right]$$

Note factor of 2 $\rightarrow = \boxed{2\times}\,\mathrm{Re}[\vec{\mu}_{12}\rho_{21}e^{-i(\omega t - kz)}]$

<u>Note</u>: The coherence ρ_{21} depends on z,t because the field depends on z,t through the envelope $\mathcal{E}(z,t)$ ➡ implicit SVEA on ρ_{21}.

<u>Note</u>: In a real, multilevel atom $\vec{\mu}_{12}$ need not be parallel to the field. However, only the part that is parallel to the field can emit radiation that interferes with it and lead to absorption and dispersion.

Maxwell-Bloch Equations

Thus, of

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The complex dipole parallel to $\vec{\varepsilon}$ is

$$\vec{P}(z,t) = \vec{\varepsilon}\, 2N \underbrace{(\vec{\mu}_{12}\cdot\vec{\varepsilon}^*)}\, \rho_{21}(z,t)\, e^{-i(\omega t - kz)}$$

$$\equiv \vec{\varepsilon}\, 2N \mu^* \rho_{21}(z,t) e^{-i(\omega t - kz)}$$

<u>Final Note</u>: Because of the RWA we have

$$\left|\frac{\partial \rho_{21}}{\partial t}\right| \ll \omega |\rho_{21}| \qquad \left|\frac{\partial^2 \rho_{21}}{\partial t^2}\right| \ll \omega \left|\frac{\partial \rho_{21}}{\partial t}\right|$$

(3) Maxwells eqs. ➡ Wave Equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}(z,t) = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\vec{P}(z,t)$$

We plug in the complex $\vec{E}(z,t)$ and $\vec{P}(z,t)$, use the SVEA conditions on the derivatives to eliminate all but the leading terms, and finally take the scalar product with $\vec{\varepsilon}$

Maxwell-Bloch Equations

Thus, of

$$\langle \vec{\mu} \rangle = \vec{\mu}_{12}\langle a_2 a_1^* \rangle + \vec{\mu}_{21}\langle a_1 a_2^* \rangle$$

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We plug in the complex $\vec{E}(z,t)$ and $\vec{P}(z,t)$, use the SVEA conditions on the derivatives to eliminate all but the leading terms, and finally take the scalar product with $\vec{\varepsilon}$ to get a scalar equation. **(Home Work Problem)**

03-18-2025

Maxwell-Bloch Equations

The complex dipole parallel to $\vec{\mathcal{E}}$ is

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$$\equiv \vec{\mathcal{E}}\, 2N \mu^* \rho_{21}(z,t)\, e^{-i(\omega t - kz)}$$

Note: Because of the RWA we have

$$\left|\frac{\partial \rho_{21}}{\partial t}\right| \ll \omega |\rho_{21}| \quad , \quad \left|\frac{\partial^2 \rho_{21}}{\partial t^2}\right| \ll \omega \left|\frac{\partial \rho_{21}}{\partial t}\right|$$

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We plug in the complex $\vec{E}(z,t)$ and $\vec{P}(z,t)$, use the SVEA conditions on the derivatives to eliminate all but the leading terms, and finally take the scalar product with $\vec{\mathcal{E}}$ to get a scalar equation. **(Home Work Problem)**

This gives us our final equation for the envelope:

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\mathcal{E}(z,t) = \frac{ik}{\varepsilon_0} N \mu^* \rho_{21}(z,t)$$

where $\mu^* = \vec{\mu}_{12}\cdot\vec{\mathcal{E}}^*$

Write ρ_{21} in terms of the Bloch variables to get the

Maxwell-Bloch Equations

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\mathcal{E}(z,t) = \frac{ik}{2\varepsilon_0} N \mu^* (u - iv)$$

$$\dot{u} = -\beta u + \mathrm{Im}[X]\, w + \Delta v$$

$$\dot{v} = -\beta v + \Delta u + \mathrm{Re}[X]\, w$$

$$\dot{w} = -\frac{1}{T_1}(1 + w) - \mathrm{Re}[X]\, v - \mathrm{Im}[X]\, u$$

Note: The Maxwell-Bloch Equations are a key result. They lead to rich physics, including absorption, gain, dispersion, solitons, lasers, and much more.

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&\dot{u}=-\beta u+\mathrm{Im}[\chi]w+\Delta v\\
&\dot{v}=-\beta v+\Delta u+\mathrm{Re}[\chi]w\\
&\dot{w}=-\frac{1}{T_1}(1+w)-\mathrm{Re}[\chi]v-\mathrm{Im}[\chi]u
\end{aligned}}$$

Note: The Maxwell-Bloch Equations are a key result. They lead to rich physics, including absorption, gain, dispersion, self-induced transparency, solitons, lasers, and much more.

Steady-State Solutions to MBE's

Steady state means that

$$\frac{1}{c}\frac{\partial\mathcal{E}}{\partial t}=0 \quad \& \quad \rho_{21}(z,t)\rightarrow\rho_{21}(z,\infty)=\frac{-i\chi/2}{\beta+i\Delta}(\rho_{22}-\rho_{11})$$

Combine with $\chi=-\vec{r}_{21}\cdot\vec{\varepsilon}\,\mathcal{E}/\hbar=\mu\mathcal{E}/\hbar$

$$\frac{\partial\mathcal{E}}{\partial z}=\frac{ik}{\varepsilon_0}N\mu^*\left(\frac{-i\mu\mathcal{E}}{2\hbar}\right)\frac{1}{\beta+i\Delta}(\rho_{22}-\rho_{11})$$

$$=\frac{kN|\mu|^2}{2\hbar\varepsilon_0}\frac{\beta-i\Delta}{\Delta^2+\beta^2}(\rho_{22}-\rho_{11})\mathcal{E}$$

We can rewrite this as

$$\boxed{\begin{aligned}
&\frac{\partial\mathcal{E}}{\partial z}=\frac{1}{2}(a-i\delta)w\mathcal{E}\\
&a=\frac{kN|\mu|^2}{2\hbar\varepsilon_0}\frac{\beta}{\Delta^2+\beta^2}=N\sigma(\Delta)\\
&\delta=\frac{kN|\mu|^2}{2\hbar\varepsilon_0}\frac{\Delta}{\Delta^2+\beta^2}=N\frac{\Delta}{\beta}\sigma(\Delta)
\end{aligned}}$$

Maxwell-Bloch Equations

Steady-State Solutions to MBE's

Steady state means that

$$\frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} = 0 \quad \& \quad \rho_{21}(z,t) \to \rho_{21}(z,\infty) = \frac{-i\chi/2}{\beta + i\Delta}(\rho_{22} - \rho_{11})$$

Combine with $\chi = -\vec{\mu}_{21} \cdot \vec{\varepsilon}\, \mathcal{E}/\hbar = \mu \mathcal{E}/\hbar$

$$\frac{\partial \mathcal{E}}{\partial z} = \frac{ik}{\varepsilon_0} N \mu^* \left(\frac{-i\mu\mathcal{E}}{2\hbar}\right)\frac{1}{\beta + i\Delta}(\rho_{22} - \rho_{11})$$

$$= \frac{kN|\mu|^2}{2\hbar\varepsilon_0}\frac{\beta - i\Delta}{\Delta^2 + \beta^2}(\rho_{22} - \rho_{11})\mathcal{E}$$

We can rewrite this as

$$\boxed{\begin{aligned} \frac{\partial \mathcal{E}}{\partial z} &= \frac{1}{2}(a - i\delta)\,\omega\, \mathcal{E} \\ a &= \frac{kN|\mu|^2}{2\hbar\varepsilon_0}\frac{\beta}{\Delta^2+\beta^2} = N\sigma(\Delta) \\ \delta &= \frac{kN|\mu|^2}{2\hbar\varepsilon_0}\frac{\Delta}{\Delta^2+\beta^2} = N\frac{\Delta}{\beta}\sigma(\Delta) \end{aligned}}$$

To compare with our classical theory of dispersion, we solve for $\mathcal{E}(z)$ and plug into eq. for a plane wave.

$$\left.\begin{aligned} &\textbf{Field:} \quad E(z) = \mathcal{E}(z)e^{ikz} \\ &\textbf{Envelope:} \quad \mathcal{E}(z) = \mathcal{E}(0)e^{\left(\frac{a\omega}{2}\right)z} e^{i\left(-\frac{\delta\omega}{2}\right)z} \end{aligned}\right\}$$

Field: $E(z) = \mathcal{E}(0)e^{\left(\frac{a\omega}{2}\right)z} e^{i\left(1 - \frac{\delta\omega}{2k}\right)kz}$

<u>Compare</u> to: $E(z) = E_0 e^{-n_I kz} e^{i n_R kz}$

Real & Imaginary Index of Refraction

$$\boxed{\begin{aligned} n_I &= -\frac{a\omega}{2k} = -\frac{N\omega}{2k}\sigma(\Delta) \\ n_R &= 1 - \frac{\delta\omega}{2k} = 1 - \frac{\Delta}{\beta}\frac{N\omega}{2k}\sigma(\Delta) \end{aligned}}$$

Analogous to results from Electron Oscillator

$$n_I(\omega) = \frac{Ne^2}{4\varepsilon_0 m_e \omega} = \frac{\beta}{\Delta^2+\beta^2}, \quad n_R(\omega) = 1 + \frac{\Delta}{\beta}n_I(\omega)$$

Maxwell-Bloch Equations

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Behavior of the Intensity

$$\frac{\partial}{\partial z}[\mathcal{E}^*\mathcal{E}] = \mathcal{E}^*\frac{\partial \mathcal{E}}{\partial z} + \frac{\partial \mathcal{E}^*}{\partial z}\mathcal{E}$$

$$= \frac{1}{2}(a-id)\omega|\mathcal{E}|^2 + \frac{1}{2}(a+id)\omega|\mathcal{E}|^2 = a\omega|\mathcal{E}|^2$$

$$\boxed{\frac{\partial I}{\partial z} = a\omega I = a(\rho_{22}-\rho_{11}) I}$$

Note that $\begin{cases} a = N\sigma(\Delta) \geq 0 \\ I(z) = I(0) e^{a(\rho_{22}-\rho_{11})z} \end{cases}$

Exp. Decay of I for $\rho_{22}-\rho_{11} < 0$

Exp. growth of I for $\rho_{22}-\rho_{11} > 0$

must be maintained by some external process

Maxwell-Bloch Equations

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Real & Imaginary Index of Refraction

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Maxwell-Bloch Equations

Behavior of the Intensity

$$\frac{\partial}{\partial z}|\mathcal{E}^*\mathcal{E}| = \mathcal{E}^*\frac{\partial \mathcal{E}}{\partial z} + \frac{\partial \mathcal{E}^*}{\partial z}\mathcal{E}$$

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Behavior of the Dispersion:

Real & Imaginary Index of Refraction

$$n_I = -\frac{a\omega}{2k} = -\frac{N\omega}{2k}\sigma(\Delta)$$

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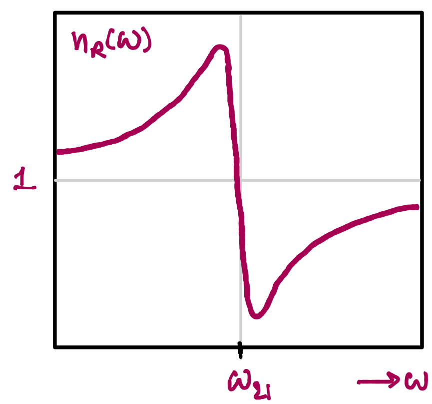
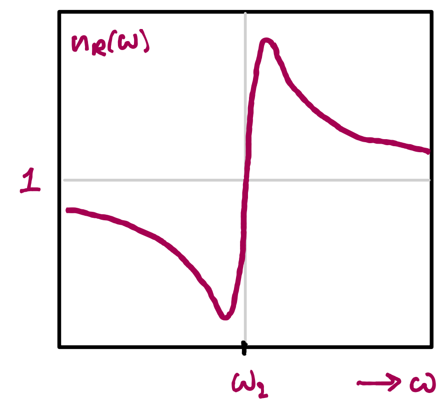
$w<1$ absorption



$w>1$ gain



Maxwell-Bloch Equations - Solitons

Self-Induced Transparency & Solitons

(*) Example of a non-trivial application of the MBE's in the context of pulse propagation (highly dynamic, non-steady state behavior.

(*) The pulse area theorem suggests a light pulse with the proper envelope will act as a 2π pulse. Thus, if the pulse is shorter than the excited state lifetime it may propagate without loss. Correct shaping may also allow propagation without changes in pulse shape.

(*) See Lecture Notes, Slusher & Gibbs 1972.

Envelope: $\mathcal{E}(z,t) = \frac{2\hbar}{\mu\tau}\,\mathrm{sech}(\xi/\tau),\ \xi = t - z/V,\ \Delta = 0$

⇨ $\chi(z,t) = \frac{2}{\tau}\mathrm{sech}(\xi/\tau),\quad \theta = \int_{-\infty}^{\infty}\chi(\xi/\tau)\,dt = 2\pi$

Self-consistent solution with the the properties of a Soliton

$$\boxed{\begin{aligned}&\mathcal{E}(z,t) = \frac{2\hbar}{\mu\tau}\,\mathrm{sech}(\xi/\tau)\\ &u(\xi/\tau) = 0\\ &v(\xi/\tau) = 2\,\mathrm{sech}(\xi/\tau)\tanh(\xi/\tau)\\ &w(\xi/\tau) = -1 + 2\,\mathrm{sech}(\xi/\tau)\end{aligned}}$$

In the SVEA version of the Wave Eq.

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\mathcal{E}(z,t) = \frac{ik}{\varepsilon_0}N\mu^*(u - iv)$$

Substitute solutions for $\mathcal{E}$, u and v to get

$$\frac{2\hbar}{\mu\tau}\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial z}\right)\mathrm{sech}\left(\frac{t - zV}{\tau}\right) =$$

$$\frac{2\hbar}{\mu\tau}\left(\frac{-1}{V\tau} + \frac{1}{c\tau}\right)\left[-\mathrm{sech}\left(\frac{t-zV}{\tau}\right)\tanh\left(\frac{t-zV}{\tau}\right)\right] =$$

$$\frac{2kN\mu^*}{\varepsilon_0}\,\mathrm{sech}\left(\frac{t-zV}{\tau}\right)\tanh\left(\frac{t-zV}{\tau}\right)$$

Solve for C/V to get

$$\frac{C}{V} = 1 + \frac{kN|\mu|^2}{2\varepsilon_0\hbar}C\tau^2 = 1 + \frac{1}{2}\alpha\beta C\tau^2$$

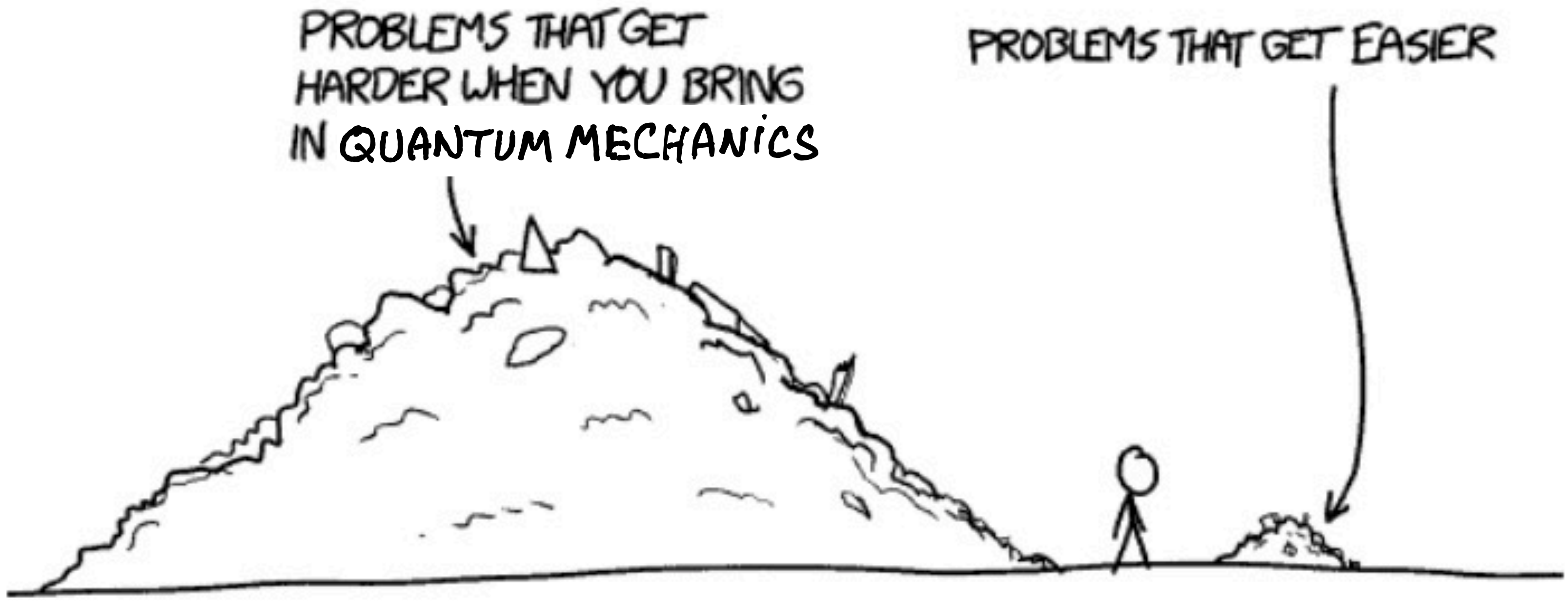
where $\alpha = \frac{kN|\mu|^2}{2\varepsilon_0\hbar\beta} = N\sigma(0)$ (on-resonance absorption coeff.)

Consider Na vapor, $\lambda = 589\,\mathrm{nm}$, $N = 10^{18}\,\mathrm{m^{-3}}$, $T \sim 0K$, **and** $\beta = 2\pi \times 4.9\,\mathrm{MHz}$ **(completely opaque on res.)**

Assuming $V \sim \frac{1}{2}C$ ⇨ $\frac{C}{V} \sim 2 = 1 + \alpha\beta C\tau^2$ ⇨ $\frac{1}{2}\alpha\beta C\tau^2 \sim 1$

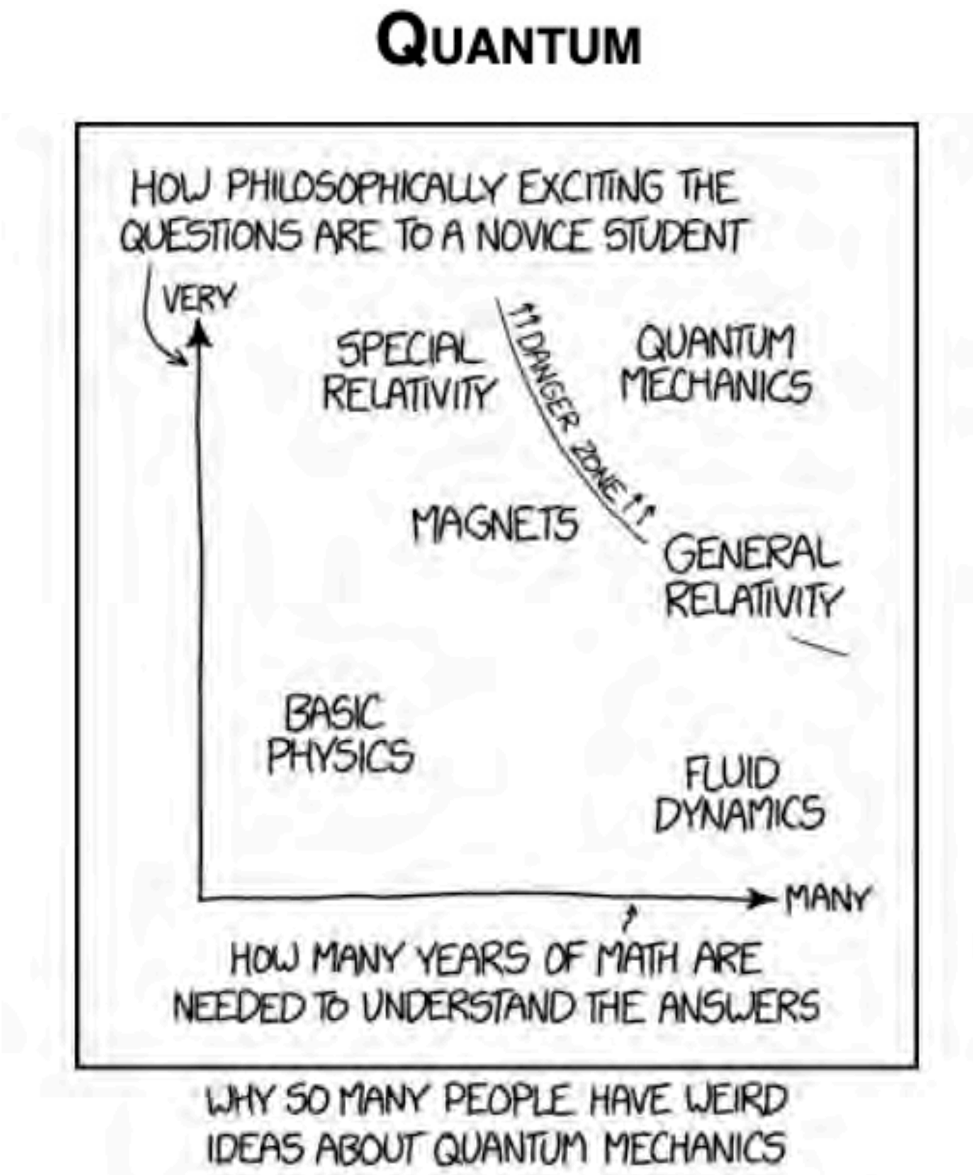
we must have $\tau \sim \sqrt{\frac{2}{\alpha\beta c}} \sim 36\,\mathrm{ps} \ll 16\,\mathrm{ns}$ **!!**

Quantum Electrodynamics – Intro to Field Theory



Source: xkcd.com

Quantum Electrodynamics – Intro to Field Theory



Quantum Electrodynamics – Intro to Field Theory

(∗) **Primary goal of OPTI 544:**
Quantum description of EM field

(∗) **Challenge: 1st semester Grad level QM (OPTI 570) does not tell how to do this**

(∗) **Warm-up: Quantum field theory for vibrations (sound) in elastic rod**

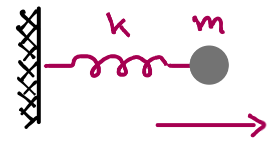
(∗) **This is in part a review of the <u>classical</u> Lagrange/Hamilton-Jacobi description of continuous systems**

(∗) **Here we present the formalism as a Cookbook Recipe for how we get from Classical to Quantum Physics**

See, e. g., Cohen-Tannoudji Vol. 2, Appendix III, Sections 1-3.

Classical Simple Harmonic Oscillator (SHO)

Particle on a spring



$\omega = \sqrt{k/m}$

osc. freq.

dynamical coordinate.

Kinetic Energy: $T = \frac{1}{2} m \dot{q}^2$

Potential Energy: $V = \frac{1}{2} k q^2 = \frac{1}{2} m \omega^2 q^2$

<u>Lagrangian</u>: $\mathcal{L} = T - V = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2$

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad \Rightarrow \quad \ddot{q} + \omega^2 q = 0$$

usual eq. of motion

Quantum Electrodynamics – Intro to Field Theory

(*) **Primary goal of OPTI 544:**
Quantum description of EM field

(*) **Challenge: 1st semester Grad level QM (OPTI 570) does not tell how to do this**

(*) **Warm-up: Quantum field theory for vibrations (sound) in elastic rod**

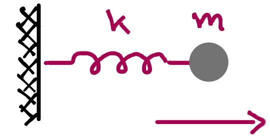
(*) **This is in part a review of the <u>classical</u> Lagrange/Hamilton-Jacobi description of continuous systems**

(*) **Here we present the formalism as a Cookbook Recipe for how we get from Classical to Quantum Physics**

See, e. g., Cohen-Tannoudji Vol. 2, Appendix III, Sections 1-3.

Classical Simple Harmonic Oscillator (SHO)

Particle on a spring



dynamical coordinate.

$\omega = \sqrt{k/m}$

osc. freq.

Kinetic Energy: $T = \frac{1}{2} m \dot{q}^2$

Potential Energy: $V = \frac{1}{2} k q^2 = \frac{1}{2} m \omega^2 q^2$

<u>Lagrangian</u>: $\mathcal{L} = T - V = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2$

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad \Rightarrow \quad \ddot{q} + \omega^2 q = 0$$

usual eq. of motion

Quantum Electrodynamics – Intro to Field Theory

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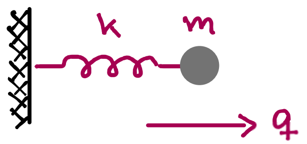

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Conjugate momentum $p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m \dot{q}$

Hamiltonian

$$\mathcal{H} = T(\dot{q} = p/m) + V(q) = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2$$

$$\left.\begin{aligned} \dot{q} &= \frac{\partial \mathcal{H}}{\partial p} = p/m \\ \dot{p} &= -\frac{\partial \mathcal{H}}{\partial q} = -m\omega^2 q \end{aligned}\right\} \Rightarrow \ddot{q} + \omega^2 q = 0$$

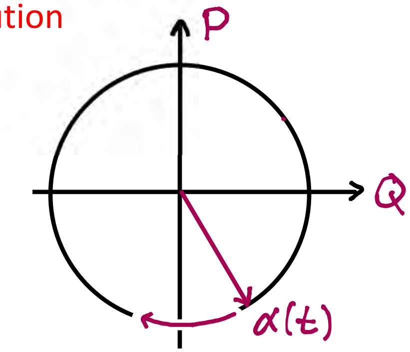
solution

Scaled variables

$Q = q/q_0, \quad P = p/p_0$

$$\alpha = Q + iP \left\{ \begin{aligned} Q &= \mathrm{Re}[\alpha] \\ P &= \mathrm{Im}[\alpha] \\ \mathcal{H} &= E_0 \alpha^* \alpha \end{aligned} \right.$$

Phase plane



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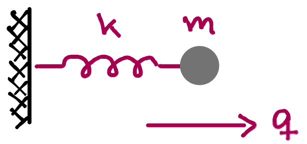
Classical Simple Harmonic Oscillator (SHO)

Particle on a spring



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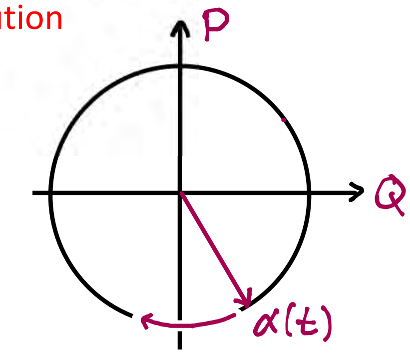

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Conjugate momentum

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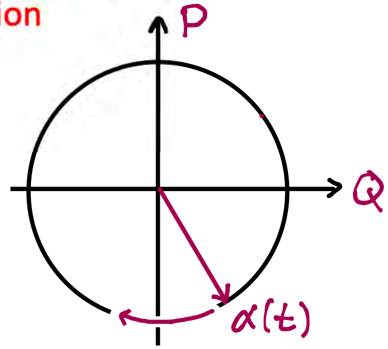

Quantum Harmonic Oscillator

Formal Quantization Procedure:

$$\boxed{q \to \hat{q}, \quad p \to \hat{p}, \quad [\hat{q}, \hat{p}] = i\hbar}$$

Choose $E_0 = \hbar\omega$ $\Rightarrow$ $q_0 = \sqrt{\frac{2\hbar}{m\omega}}$, $p_0 = \sqrt{2m\hbar\omega}$

natural scale

$$\alpha \to \hat{a} = \hat{Q} + i\hat{P} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} + i\frac{\hat{p}}{m\omega}\right)$$

$$[\hat{a}, \hat{a}^+] = 1$$

Rewrite:

$$\hat{H} = \hbar\omega(\hat{Q}^2 + \hat{P}^2) = \hbar\omega(\hat{a}^+\hat{a} + 1/2)$$

$\hat{N} = \hat{a}^+\hat{a}$ **(number operator)**

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$\hat{N} = \hat{a}^+\hat{a}$ (number operator)

Commutator $[\hat{H}, \hat{N}] = 0$

$\Rightarrow$ **joint energy/number states** $|n\rangle$

$$\hat{H}|n\rangle = \hbar\omega(n + 1/2)|n\rangle$$

$$\hat{N}|n\rangle = n|n\rangle$$

Commutators

$$\left.\begin{aligned} [\hat{N}, \hat{a}^+] &= \hat{a}^+ \\ [\hat{N}, \hat{a}] &= -\hat{a} \end{aligned}\right\} \Rightarrow \begin{aligned} \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \\ \hat{a}^+|n\rangle &= \sqrt{n+1}|n+1\rangle \\ \hat{a}|0\rangle &= 0 \end{aligned}$$

Generating excited states

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^+)^n|0\rangle}$$

Quantum Electrodynamics – Intro to Field Theory

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Expectation values for $\hat{q}$ and $\hat{p}$ in number states

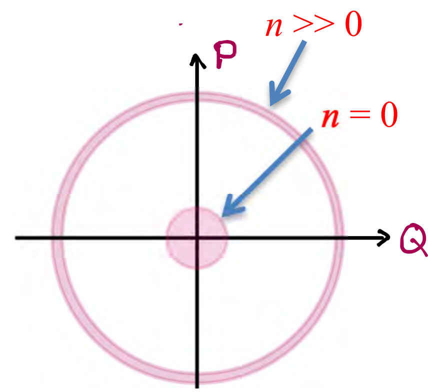
$$\langle n|\hat{q}|n\rangle = \langle n|\hat{p}|n\rangle = 0$$

$$\langle n|\hat{q}^2|n\rangle = \frac{q_0^2}{2}(n+1/2) \neq 0$$

$$\langle n|\hat{p}^2|n\rangle = \frac{p_0^2}{2}(n+1/2) \neq 0$$

$$\Delta q\,\Delta p = \frac{q_0 p_0}{2}(n+1/2) = \hbar(n+1/2)$$

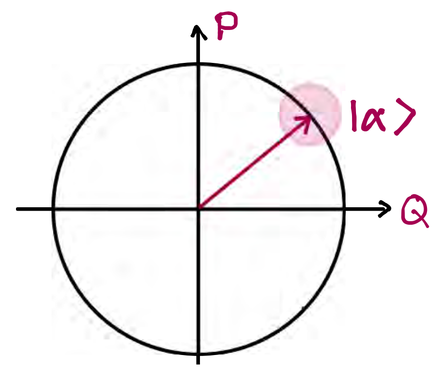
Phase space visualization of number states



Quasi-classical (coherent) state

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_i \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$\Delta q\,\Delta p = \hbar/2, \quad \Delta Q = \Delta P$$



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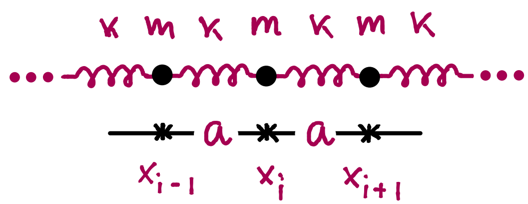
Lagrange formulation of 1D Scalar Field

"vibronium"



Configuration space = $\{x_i\}$ (set of *N* osc. positions)

$$T = \sum_{i=1} \tfrac{1}{2} m \dot{x}_i^2 , \quad V = \sum_{i=1}^{N} \tfrac{1}{2} \kappa (x_{i+1} - x_i)^2$$

Lagrangian, equations of motion

Continuum limit ➡ **Elastic rod**

$N \to \infty$ $\quad m/a \to \mu$ ← linear mass density

$a \to dx$ $\quad \kappa a \to Y$ ← Youngs modulus

$\{x_i\} \to y(x)$ ← displacement field (sound)

Rewrite

$$\boxed{\begin{aligned} T &= \lim_{N\to\infty} \sum_{i=1}^{N} a \tfrac{1}{2} \left(\tfrac{m}{a}\right) \dot{x}_i^2 = \int dx \tfrac{1}{2} \mu \left(\frac{\partial y}{\partial t}\right)^2 \\ V &= \lim_{N\to\infty} \sum_{i=1}^{N} a \tfrac{1}{2} \kappa a \left(\frac{x_{i+1} - x_i}{a}\right)^2 = \int dx \tfrac{1}{2} Y \left(\frac{\partial y}{\partial x}\right)^2 \end{aligned}}$$

Lagrangian:

$$\boxed{\mathcal{L} = T - V = \int dx \tfrac{1}{2} \mu \left(\frac{\partial y}{\partial t}\right)^2 - \int dx \tfrac{1}{2} Y \left(\frac{\partial y}{\partial x}\right)^2}$$

Notes, Homework ➡ **Scalar wave equation**

$$\boxed{\frac{\partial^2 y}{\partial t^2} - \frac{Y}{\mu} \frac{\partial y^2}{\partial x^2} = 0}$$

– Not yet ready for Quantization –

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Rewrite

$$T = \lim_{N\to\infty} \sum_{i=1}^{N} a \tfrac{1}{2}\left(\frac{m}{a}\right)\dot{x}_i^2 = \int dx \tfrac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2$$

$$V = \lim_{N\to\infty} \sum_{i=1}^{N} a \tfrac{1}{2}\kappa a\left(\frac{x_{i+1}-x_i}{a}\right)^2 = \int dx \tfrac{1}{2} Y\left(\frac{\partial y}{\partial x}\right)^2$$

Lagrangian:

$$\mathcal{L} = T - V = \int dx \tfrac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 - \int dx \tfrac{1}{2} Y\left(\frac{\partial y}{\partial x}\right)^2$$

Notes, Homework ⇨ **Scalar wave equation**

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Normal Mode Decomposition

Field in cavity:

Solutions to wave eq.

Let $y(x,t) = g(t)u(x) = g_0 e^{i\omega t} u(x)$ ⇨

$$\ddot{y} - v^2 y'' = -\omega^2 g(t)u(x) - v^2 g(t)u''(x) = 0$$

$$u''(x) = -k^2 u(x), \quad k = \omega/v$$

Solutions in cavity:

$$u_k(x) = \sqrt{\frac{2}{L}}\sin(kx), \quad k = \frac{n\pi}{L}$$

These standing waves are a set of Normal Modes

Quantum Electrodynamics – Intro to Field Theory

Normal Mode Decomposition

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These modes are orthonormal and complete

$$\boxed{y(x,t) = \sqrt{L} \sum_k q_k(t) u_k(x)}$$

Normal mode expansion of $y(x,t)$ **in basis** $u_k(x)$

Lagrangian for the acoustic field:

$$T = \int dx \frac{1}{2} \mu \left(\frac{\partial y}{\partial t}\right)^2 = \sum_{k,k'} \frac{1}{2} \underbrace{\mu L}_{M} \dot{q}_k \dot{q}_{k'} \underbrace{\int dx\, u_k(x) u_{k'}(x)}_{\delta_{kk'}}$$

$$= \sum_k \frac{1}{2} M \dot{q}_k^2$$

$$V = \int dx \frac{1}{2} Y \left(\frac{\partial y}{\partial x}\right)^2 = \sum_{kk'} \frac{1}{2} Y L q_k q_{k'} \int dx \left(\frac{\partial u_k}{\partial x}\right)\left(\frac{\partial u_{k'}}{\partial x}\right)$$

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Quantum Electrodynamics – Intro to Field Theory

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The rest now follows from the Lagrangian

$$\mathcal{L} = T - V = \sum_k \left(\frac{1}{2} M \dot{q}_k^2 - \frac{1}{2} M \omega_k^2 q_k^2 \right) = \sum_k \mathcal{L}_k$$

Canonical Momentum

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = M \dot{q}_k$$

Hamiltonian

$$\mathcal{H}(\{p_k, q_k\}) = T + V = \sum_k \left(\frac{p_k^2}{2M} + \frac{1}{2} M \omega_k^2 q_k^2 \right)$$

(collection of SHO's, one for each normal mode)

Following the standard recipe…

$$E_{0,k} = \hbar\omega_k \;,\; q_{0,k} = \sqrt{2\hbar / M\omega_k} \;,\; p_{0,k} = \sqrt{2M\hbar\omega_k}$$

$$Q_k = q_k / q_{0,k} \;,\; P_k = p_k / p_{0,k} \;,\; \alpha_k = Q_k + i P_k$$

… we get solutions

$$\alpha_k(t) = Q_k(t) + i P_k(t) = \alpha_k(0)\, e^{-i\omega_k t}$$

This finally gives us

$$\mathcal{H} = \sum_k \hbar\omega_k \left(Q_k^2 + P_k^2 \right) = \sum_k \hbar\omega_k \alpha_k^* \alpha_k$$

$$y(x,t) = \sqrt{L} \sum_k q_k(t) u_k(x)$$

$$= \frac{1}{2} \sum_k \sqrt{L q_{0,k}^2} \left(\alpha_k(t) u_k(x) + \alpha_k^*(t) u_k^*(x) \right)$$