$$T_2 \leq 2T_1 \tag{8.2.20}$$

between the transverse and longitudinal lifetimes.

8.3 MAXWELL–BLOCH EQUATIONS

The interaction of light and atoms has two sides. As we emphasized in Chapter 1, this interaction should ideally be dealt with self-consistently. We have not yet done this in the quantum framework, having taken the field to be a fixed monochromatic wave for the most part.

Maxwell's wave equation (2.1.13) remains valid in quantum mechanics. Given a source polarization, even one described quantum-mechanically, we can solve the wave equation to find $\mathbf{E}(\mathbf{r}, t)$. As usual, we will assume that only the z coordinate will be significant, and that the field will be almost monochromatic. That is, we will assume that $\mathbf{E}$ can be conveniently written

$$\mathbf{E}(\mathbf{r}, t) = \hat{\varepsilon}\mathcal{E}(z, t)\, e^{-i(\omega t - kz)} \tag{8.3.1}$$

where the real part is understood to be the physical electric field, and where $\mathcal{E}(z, t)$ is the unknown complex amplitude to be determined. An additional assumption implied by the form assumed in (8.3.1) is that the amplitude $\mathcal{E}(z, t)$ varies slowly compared with the carrier wave $e^{-i(\omega t - kz)}$. This justifies inequalities such as

$$\left|\frac{\partial \mathcal{E}}{\partial z}\right| \ll k|\mathcal{E}|$$
$$\left|\frac{\partial^2 \mathcal{E}}{\partial z^2}\right| \ll k\left|\frac{\partial \mathcal{E}}{\partial z}\right|$$
$$\left|\frac{\partial \mathcal{E}}{\partial t}\right| \ll \omega|\mathcal{E}| \tag{8.3.2}$$

In physical terms these inequalities state that $\mathcal{E}(z, t)$ represents a smooth enough pulse in both space and time. This restriction is not severe, since it would be violated only if $\mathcal{E}(z, t)$ represented a pulse shorter than a few optical periods ($\sim 10^{-15}$ sec) in time or a few wavelengths ($\sim 1\ \mu$m) in space.

In parallel with (8.3.1) and (8.3.2), we make similar assumptions about the polarization density arising from whatever atoms are present. The semiclassical theory uses the quantum expectation value for the polarization density, namely $Ne\langle \mathbf{r} \rangle$. In complex form analogous to (8.3.1) we have

$$\begin{aligned} \mathbf{P}(z, t) &= 2Ne\mathbf{r}_{12}\, a_1^* a_2 \\ &= 2Ne\mathbf{r}_{12}\rho_{21}(z, t)e^{-i(\omega t - kz)} \end{aligned} \tag{8.3.3}$$

where the real part is the physical polarization. We have used (6.4.3), (6.3.12), and (6.5.1). The slowly varying character of $\mathcal{E}(z, t)$ is also imputed to ρ_{21}:

$$\left|\frac{\partial \rho_{21}}{\partial t}\right| \ll \omega|\rho_{21}|$$
$$\left|\frac{\partial^2 \rho_{21}}{\partial t^2}\right| \ll \omega\left|\frac{\partial \rho_{21}}{\partial t}\right| \tag{8.3.4}$$

etc.

The wave equation for one spatial propagation direction (the z direction) is

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(z, t) = \frac{1}{\epsilon_0 c^2}\frac{\partial^2}{\partial t^2}\mathbf{P}(z, t) \tag{8.3.5}$$

After substituting (8.3.1) and (8.3.3) into (8.3.5), and making use of (8.3.2) and (8.3.4), we keep the largest terms (lowest-order derivatives) on each side. After projecting both sides on $\hat{\varepsilon}^*$ and using $\hat{\varepsilon}^* \cdot \hat{\varepsilon} = 1$ we obtain the wave equation

$$\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial ct}\right)\mathcal{E}(z, t) = \frac{ik}{\epsilon_0} N\mu^*\rho_{21}(z, t) \tag{8.3.6}$$

where we have used the convenient abbreviation [recall (7.2.5)]

$$\mu^* = e(r_{12} \cdot \hat{\varepsilon}^*) \tag{8.3.7}$$

for the projection of the transition dipole moment on the direction of polarization. When the relations (8.2.1) for the Bloch variables u and v are introduced, we find

$$\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial ct}\right)\mathcal{E}(z, t) = \frac{ik}{2\epsilon_0} N\mu^*(u - iv) \tag{8.3.8}$$

Both (8.3.6) and (8.3.8) are known as the reduced wave equation, or the wave equation in the slowly-varying-envelope approximation. Equations (8.2.18) and (8.3.8) together are said to be the coupled *Maxwell–Bloch equations* (Problem 8.4).

With these equations we have a quantum theory that can be treated self-consistently. That is, the coupled Maxwell–Bloch equations allow the atoms and the field to influence each other mutually, and the theory treats this mutual interaction at a fundamental level (Figure 8.4). We already saw in Chapter 1 an earlier example of such a mutual interaction [recall Figure 1.15], but there the theory was completely empirical. We now reexamine some earlier results, including those of Chapter 1, from our present, more satisfactory foundation.

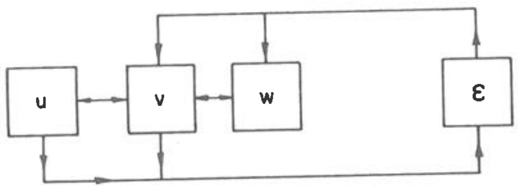

Figure 8.4 The mutual interactions embodied in the Maxwell-Bloch equations. The coupling is much more intricate than in the conventional rate-equation theory illustrated in Figure 1.15.

8.4 LINEAR ABSORPTION AND AMPLIFICATION

First we determine what effect other atoms, e.g., the collision partners of our two-level atoms, have on the Maxwell equation. These atoms also have dipole moments and give rise to an added polarization density. Thus we add the term

$$\frac{ik}{\epsilon_0}\bar{N}\,\bar{\mu}^*\,\bar{\rho}_{21}(z,t) \tag{8.4.1}$$

to the right side of (8.3.6), where the overbars denote background-atom parameters. These atoms are far from resonance and come to steady state extremely quickly, so we can use the adiabatic result (7.2.1) for $\bar{\rho}_{21}$. We can safely assume that the background atoms are at most only very slightly excited, so that $\bar{\rho}_{22} = 0$ and $\bar{\rho}_{11} = 1$. Thus we have

$$\begin{aligned}\bar{\rho}_{21} &= \frac{i\bar{\chi}/2}{\bar{\beta} + i\bar{\Delta}}\\ &= \frac{i(\bar{\chi}/2)(\bar{\beta} - i\bar{\Delta})}{\bar{\Delta}^2 + \bar{\beta}^2}\end{aligned} \tag{8.4.2}$$

where $\bar{\chi} = \bar{\mu}\mathcal{E}/\hbar$. This expression for $\bar{\rho}_{21}$ gives

$$\begin{aligned}\frac{ik}{\epsilon_0}\bar{N}\,\bar{\mu}^*\,\bar{\rho}_{21} &= \frac{\bar{N}\,\bar{\mu}^*\,\omega\,\bar{\mu}\,\mathcal{E}}{2\epsilon_0 c}\,\frac{i\bar{\Delta} - \bar{\beta}}{\hbar\;(\bar{\Delta}^2 + \bar{\beta}^2)}\\ &= -\frac{1}{2}\left(\bar{a} - i\bar{\delta}\right)\mathcal{E}\end{aligned} \tag{8.4.3}$$

Here we have defined

$$\bar{a} = \frac{\bar{N}\,|\bar{\mu}|^2\omega}{\epsilon_0\hbar c}\,\frac{\bar{\beta}}{\bar{\Delta}^2 + \bar{\beta}^2} \tag{8.4.4}$$

$$\bar{\delta} = (\bar{\Delta}/\bar{\beta})\,\bar{a} = \frac{\bar{N}\,|\bar{\mu}|^2\omega}{\epsilon_0\hbar c}\,\frac{\bar{\Delta}}{\bar{\Delta}^2 + \bar{\beta}^2} \tag{8.4.5}$$

where $\bar{a}$ is seen by comparison with (7.4.8) to be the extinction coefficient of the background atoms. It is also the imaginary part of the index of refraction for light transmitted through the background atoms alone. [Recall (3.4.11) with the oscillator strength f included.] Similarly $\bar{\delta}$ can be recognized by comparison with (3.4.9) to correspond to the background correction to the real part of the index of refraction.

The effect of the background atoms (collision partners, etc.) on the slowly varying Maxwell equation (8.3.6) is therefore simply to add two terms to the left side:

$$\left(\frac{\partial}{\partial z} + \frac{\bar{a}}{2} - \frac{i\bar{\delta}}{2} + \frac{\partial}{\partial ct}\right)\mathcal{E}(z,t) = \frac{ik}{\epsilon_0}N\mu^*\rho_{21}(z,t) \tag{8.4.6}$$

Now consider the Maxwell–Bloch equations in steady state. We discard the time derivative in (8.4.6) and use the quasisteady solution (7.2.1b) for ρ_{21}. All of the steps from (8.3.8) to (8.4.6) apply as well to ρ_{21} as to $\bar{\rho}_{21}$, except that we cannot assume $\rho_{22} = 0$ and $\rho_{11} = 1$ for on-resonant atoms. This minor distinction is easily accounted for, and we can write (8.4.6) as

$$\left(\frac{\partial}{\partial z} + \frac{\xi - i\eta}{2}\right)\mathcal{E}(z) = 0 \tag{8.4.7}$$

where we have introduced the temporary abbreviation

$$\frac{\xi - i\eta}{2} = \frac{1}{2}\left(\bar{a} - i\bar{\delta} + (a - i\delta)\,(\rho_{11} - \rho_{22})\right) \tag{8.4.8}$$

Let us multiply (8.4.7) by $\mathcal{E}^*$ and add the complex conjugate equation to get

$$\mathcal{E}^*\left(\frac{\partial}{\partial z} + \frac{\xi - i\eta}{2}\right)\mathcal{E} + \mathcal{E}\left(\frac{\partial}{\partial z} + \frac{\xi + i\eta}{2}\right)\mathcal{E}^* = 0 \tag{8.4.9}$$

Since $\partial|\mathcal{E}|^2/\partial z = \mathcal{E}^*(\partial\mathcal{E}/\partial z) + \text{c.c.}$, this is the same as

$$\left(\frac{\partial}{\partial z} + \xi\right)|\mathcal{E}|^2 = 0 \tag{8.4.10}$$

and since $|\mathcal{E}|^2$ is proportional to I, we have

$$\frac{\partial I}{\partial z} = -\xi I \tag{8.4.11}$$

Equation (8.4.11) is the same as Eq. (2.6.14) and has the same exponential decay solution

$$I(z) = I(0)\, e^{-a_s z} \tag{8.4.12}$$

as was given in (2.6.15), if we identify ξ with the extinction coefficient a_s:

$$a_s \leftrightarrow \xi = \bar{a} + a(\rho_{11} - \rho_{22}) \tag{8.4.13}$$

In fact, we find much more than a simple identification of coefficients. From the Maxwell-Bloch result (8.4.11) we can draw two conclusions about light propagation in a medium of atoms both near to resonance and far off resonance (background atoms):

i. If the resonant atoms are all in their ground states ($\rho_{11} \approx 1$, $\rho_{22} \approx 0$), then the classical law of exponential extinction is valid, and $a_s = \bar{a} + a$. That is, both on-resonance and off-resonance atoms contribute alike to the attenuation of the field.

ii. If the nearly resonant atoms are in their excited states ($\rho_{11} \approx 0$, $\rho_{22} \approx 1$), then the solution is still exponential in form but with $a_s = \bar{a} - a$. It is quite possible that $a >> \bar{a}$, since the detuning $\bar{\Delta}$ appearing in the denominator of the expression for $\bar{a}$ in (8.4.4) is by assumption very large, and the detuning Δ in the corresponding expression for a is small or even zero. Under these conditions *a_s is negative, and describes not attenuation but amplification* (Figure 8.5).

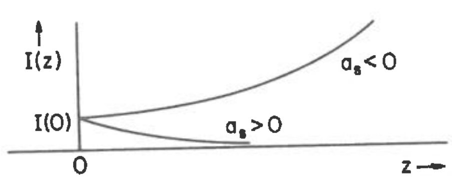

Figure 8.5 Amplification or attenuation of an incident pulse.

It is clear that in case (ii) the the possibility of laser action is foretold. Equation (8.4.13) also gives clear instructions on how to go about obtaining the right conditions for it. One need only (!) ensure that the near-resonant atoms are sufficiently excited by some means, so that

$$\rho_{22} - \rho_{11} > \bar{a}/a \tag{8.4.14}$$

All formulas for critical or threshold inversion governing laser action derive from the principles leading to (8.4.14). Mirror losses can, in practice, be more significant than attenuation due to absorption or scattering in the laser medium, but they lead to similar results, as we will find in Chapter 10. The methods for achieving threshold inversion are many. They vary greatly with type of laser, and are discussed in Chapter 13.

8.5 SEMICLASSICAL LASER THEORY

Laser action based on inversion is a quantum effect, as (8.4.14) suggests. The need for a positive inversion cannot be satisfied in a classical system, for which the concept of inversion does not exist. We present the elements of a quantum-mechanical laser theory in this section. It will not be a full theory for many reasons, but it will be complete enough to correct the flaws in the classical theory of the laser, given in Section 3.5.

We must first recall the expression given in (1.3.1) and (3.5.2) for the electric field in a laser cavity. The allowed wave vectors and mode functions are determined by the cavity length:

$$k_m = m\pi/L \tag{8.5.1}$$

so we have $\mathbf{E}(z, t) = \Sigma_m \mathbf{E}_m(z, t)$, where

$$\mathbf{E}_m = \hat{\boldsymbol{\varepsilon}}_m\, \mathcal{E}_m(t) \sin k_m z\, e^{-i\omega t} \tag{8.5.2}$$

is the electric field of the mth mode of the cavity. Note that the complex mode amplitude $\mathcal{E}_m(t)$ does not depend on z, since the cavity mode function $\sin k_m z$ is assumed to express the z dependence fully. The frequency ω of laser oscillation is not known initially, although it will be close to one of the cavity mode frequencies: $\omega \approx \omega_m$.

It is convenient to express the polarization's z dependence in terms of cavity mode functions as well. That is, we will write $\mathbf{P}(z, t) = \Sigma_m \mathbf{P}_m(z, t)$, where

$$\mathbf{P}_m = 2Ne\mathbf{r}_{12}\rho_{21}^{(m)}(z, t) \sin k_m z\, e^{-i\omega t} \tag{8.5.3}$$