OPTI 544 Solution Set 4, Spring 2024

Problem I

In steady state the Density Matrix Equations reduce to

(i) $\dot{\rho}_{11} = A_{21}\rho_{22} - \frac{i}{2}\left(\chi\rho_{12} - \chi^*\rho_{21}\right) = 0$

(ii) $\dot{\rho}_{22} = -A_{21}\,\rho_{22} + \frac{i}{2}\left(\chi\rho_{12} - \chi^*\rho_{21}\right) = 0$

(iii) $\dot{\rho}_{12} = -\left(\beta - i\Delta\right)\rho_{12} + i\frac{\chi^*}{2}\left(\rho_{22} - \rho_{11}\right) = \dot{\rho}_{21}^* = 0$

We start by solving for the coherences in Equation (iii)

$$\rho_{12} = \frac{i\frac{\chi^*}{2}\left(\rho_{22} - \rho_{11}\right)}{\beta - i\Delta} = i\frac{\chi^*}{2}\frac{\beta + i\Delta}{\Delta^2 + \beta^2}\left(\rho_{22} - \rho_{11}\right), \qquad \rho_{21} = -i\frac{\chi^*}{2}\frac{\beta - i\Delta}{\Delta^2 + \beta^2}\left(\rho_{22} - \rho_{11}\right)$$

From this we get $\chi\rho_{12} - \chi^*\rho_{21} = \frac{i|\chi|^2\beta}{\Delta^2 + \beta^2}\left(\rho_{22} - \rho_{11}\right)$

Substituting in Equation (ii), using $\rho_{22} - \rho_{11} = 2\rho_{22} - 1$, and solving for ρ_{22}, we get

$$\rho_{22} = \frac{i}{2A_{21}}\frac{i|\chi|^2\beta}{\Delta^2 + \beta^2}\left(2\rho_{22} - 1\right) \Rightarrow \left(1 + \frac{|\chi|^2\beta / A_{21}}{\Delta^2 + \beta^2}\right)\rho_{22} = \frac{1}{2}\frac{|\chi|^2\beta / A_{21}}{\Delta^2 + \beta^2}$$

$$\Rightarrow \quad \rho_{22} = \frac{|\chi|^2\beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2\beta / A_{21}}$$

Next,

$$\rho_{11} = 1 - \rho_{22} = 1 - \frac{|\chi|^2\beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2\beta / A_{21}} = \frac{\Delta^2 + \beta^2 + |\chi|^2\beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2\beta / A_{21}}$$

Sanity check: $\rho_{11} + \rho_{22} = 1$. With that we have the steady state solutions

$$\rho_{11}(\infty) = \frac{\Delta^2 + \beta^2 + |\chi|^2\beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2\beta / A_{21}} \qquad \rho_{22}(\infty) = \frac{|\chi|^2\beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2\beta / A_{21}}$$

$$\rho_{12}(\infty) = i\frac{\chi^*}{2}\frac{\beta + i\Delta}{\Delta^2 + \beta^2}\left(\rho_{22} - \rho_{11}\right) \qquad \rho_{21}(\infty) = -i\frac{\chi}{2}\frac{\beta - i\Delta}{\Delta^2 + \beta^2}\left(\rho_{22} - \rho_{11}\right)$$

Problem II

(a) We start from $\sigma(0)=\dfrac{3\lambda^2}{2\pi}$ (no collisions, polarized driving field).

Then, per definition and setting $A_{21}=1/T$ where $T=27.0\,\text{ns}$ we have

$$I_{sat}=\frac{\hbar\omega A_{21}}{2\sigma(0)}=\frac{2\pi^2\hbar c}{3\lambda^3 T}=\frac{2\pi^2\times1.05\times10^{-34}\,\text{Js}\times3\times10^8\,\text{m/s}}{3(780\times10^{-9}\,\text{m})\times27\times10^{-9}\,\text{s}}=16.2\,\text{W/m}^2=\underline{1.62\,\text{mW/cm}^2}$$

(b) We have $R_{12}=\dfrac{|\chi|^2\beta/2}{\Delta^2+\beta^2}=\sigma(\Delta)\Phi=\sigma(0)\dfrac{\beta^2}{\Delta^2+\beta^2}\dfrac{I}{\hbar\omega}$

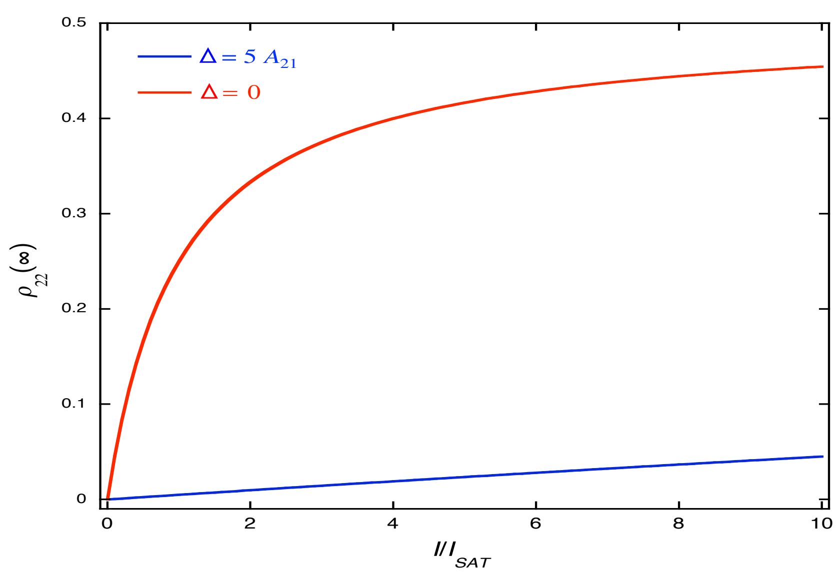
$$\Rightarrow |\chi|^2=\frac{2\beta}{\hbar\omega}\sigma(0)I=\frac{A_{21}^2}{\hbar\omega}\frac{\hbar\omega}{I_{sat}}I \quad\Rightarrow\quad \boxed{\frac{|\chi|^2}{A_{21}^2}=\frac{I}{2I_{sat}}}$$

Where we have used $2\beta=A_{21}$, $I_{sat}=\dfrac{\hbar\omega A_{21}}{2\sigma(0)}$

(c) We found in a previous Homework Problem that $\rho_{22}(\infty)=\dfrac{|\chi|^2\beta/2A_{21}}{\Delta^2+\beta^2+|\chi|^2\beta/A_{21}}$

Substitute $\dfrac{|\chi|^2}{A_{21}^2}=\dfrac{I}{2I_{sat}}$ and set $2\beta=A_{21}$ $\Rightarrow$

$$\boxed{\rho_{22}(\infty)=\frac{I/2I_{sat}}{(\Delta/\beta)^2+1+I/I_{sat}}}$$



Problem III

(a) In steady state the excited state population is (setting $2\beta = A_{21}$, $\Delta = 0$, and using the expression from the class notes)

$$\rho_{22} = \frac{|\chi|^2/4}{|\chi|^2/2 + A_{21}^2/4} = \frac{1}{2}\frac{\tilde{I}}{\tilde{I}+1},$$

where we have used $\frac{2|\chi|^2}{A_{21}^2} = \frac{I}{I_{sat}} = \tilde{I}$. Now $\Delta N = N(2\rho_{22} - 1)$

$$\boxed{\Delta N = N\left[\frac{\tilde{I}}{\tilde{I}+1} - 1\right] = -\frac{N}{\tilde{I}+1}}$$

(b) Number of stimulated emission events: $\sigma\Phi N_2$. Number of absorption evens: $\sigma\Phi N_1$

Change in photon flux: $\boxed{d\Phi = \sigma\Phi(N_2 - N_1)dz = \sigma\Phi\Delta N dz}$

(c) Dividing by Φ_{sat} and using $\tilde{I} = \frac{\Phi}{\Phi_{sat}}$ we get $\boxed{\frac{d}{dz}\tilde{I} = \sigma\tilde{I}\Delta N = -\frac{3\lambda^2}{2\pi}N\frac{\tilde{I}}{\tilde{I}+1}}$

The general solution to this equation cannot be written in terms of simple functions, so in that case one must resort to numerics. However, approximate solutions are easy to find in the limits $I << I_{sat}$ and $I >> I_{sat}$.

For $I << I_{sat}$ we have $\frac{d}{dz}\tilde{I} = \sigma\tilde{I}\Delta N = -\frac{3\lambda^2 N}{2\pi}\tilde{I} \Rightarrow \tilde{I}(z) = \tilde{I}(0)e^{-\frac{3\lambda^2 N}{2\pi}z}$

For $I >> I_{sat}$ we have $\frac{d}{dz}\tilde{I} = -\frac{3\lambda^2 N}{2\pi} \Rightarrow \tilde{I}(z) = \tilde{I}(0)(1 - \frac{3\lambda^2 N}{2\pi}z)$

(d) For the signal beam alone. $T_{off} = 0.01 = e^{-\frac{3\lambda^2 N}{2\pi}z}$

$$\Rightarrow NL = -\frac{3\lambda^2}{2\pi}\ln(0.01) = -\frac{2\pi}{3\times(1\times10^{-6}\text{m})^2}\ln(0.01) = \underline{9.65\times10^{12}\ /\text{m}^2}$$

(e) If $\tilde{I}_S << \tilde{I}_C$ then ΔN is determined solely by $\tilde{I}_C$. Thus, if $\tilde{I}_C$ is large enough to remain approximately constant along the optical path, we have a constant ΔN that is independent of $\tilde{I}_S$. In that case

$$\frac{d\tilde{I}_S}{dz} = \sigma(\Delta=0)\Delta N\,\tilde{I}_S = -\frac{3\lambda^2 N}{2\pi}\frac{\tilde{I}_S}{\tilde{I}_C+1} \Rightarrow \tilde{I}_S(z)=\tilde{I}_S(0)\,e^{-\frac{3\lambda^2}{2\pi}\frac{Nz}{\tilde{I}_C+1}}$$

We solve for the desired transmission,

$$T_{on} = 0.99 = e^{-\frac{3\lambda^2}{2\pi}\frac{Nz}{\tilde{I}_C+1}} \Rightarrow \tilde{I}_C \approx -\frac{3\lambda^2 N}{2\pi\ln(0.99)}-1 = \frac{\ln(0.01)}{\ln(0.99)}-1 = \underline{\underline{457.2}}$$

(f) We have $\tilde{I}_C >> \tilde{I}_S$ so ΔN depends only on $\tilde{I}_C$. Also we find for the transmission of the control beam that

$$T_C = 1-\frac{3\lambda^2}{2\pi}\frac{NL}{\tilde{I}_C(0)} = 1+\frac{\ln(0.01)}{457.2} = 0.9899 \approx 1$$

This means the intensity of the control beam is constant and much larger than that of the signal beam along the entire optical path.