Density Matrix Description of 2-Level Atoms

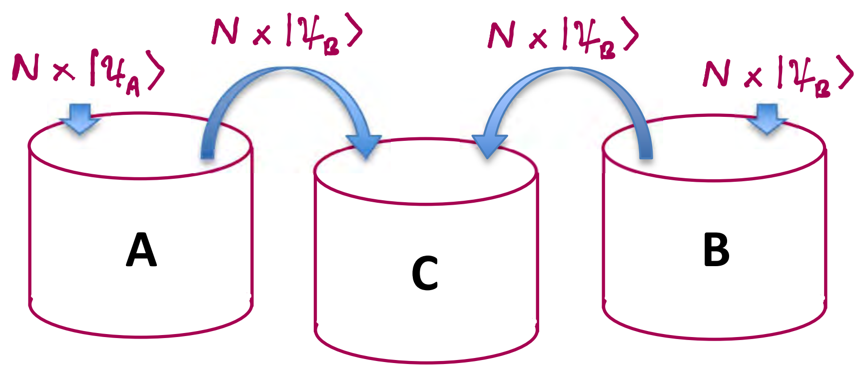
A cooks recipe – interpretations of ρ

Step 1 Add N atoms in state $|\psi_A\rangle$ to bucket A

Add N atoms in state $|\psi_B\rangle$ to bucket B

We now have two ensembles, each of which consist of N atoms in a known pure state

Step 2 Add buckets A and B to bucket C and stir.



Pick an atom from C

Which is Correct?

The atom is in a pure state but we don't know if it is in $|\psi_A\rangle$ or $|\psi_B\rangle$

The atom is in a mixed state

$$\rho = \frac{1}{2}|\psi_A\rangle\langle\psi_A| + \frac{1}{2}|\psi_B\rangle\langle\psi_B|$$

There is no difference!

The two interpretations lead to identical predictions for any measurement we can do on atoms from C

Quantum Mechanics:

If two descriptions lead to identical predictions for observable outcomes then they are <u>identical</u>

Loosely, (i) is a *frequentist view*

(ii) is a *Bayesian view*

Quantum Bayesianism

Quantum States are States of Knowledge (subjective)

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(*) Note: The notation $\langle \cdot \rangle_k$ is used on the following pages to indicate an ensemble average.

More about the Density Matrix

In the orthonormal basis $\{|u_j\rangle\}$ the elements of a pure density matrix are $\langle u_n|\rho|u_p\rangle = c_n c_p^*$. For a mixed state, $\rho = \sum_k p_k \rho_k$, we have $\rho_{np} = \sum_k p_k c_n^{(k)} (c_p^{(k)})^*$.

Here and elsewhere, the index k indicates members of the ensemble that are distinct due to, e. g. , different preparation or history.

Populations: (real-valued) $\rho_{nn} = \sum p_k c_n^{(k)} c_n^{(k)*} = \sum_k p_k |c_n^{(k)}|^2$

Single system: Prob of finding state $|u_n\rangle$

Ensemble: $|u_n\rangle$ occurs with freq. ρ_{nn}

Coherences: (complex-valued) $\rho_{np} = \sum_k p_k c_n^{(k)} (c_p^{(k)})^*$

Note: Defining $c_q = |c_q| e^{i\theta_q}$ we have

$$(*)\ \langle c_n^{(k)} c_p^{(k)*} \rangle_k = \langle |c_n^{(k)}||c_p^{(k)}| e^{i(\theta_n^{(k)} - \theta_p^{(k)})} \rangle_k \le \langle |c_n^{(k)}||c_p^{(k)}| \rangle_k$$

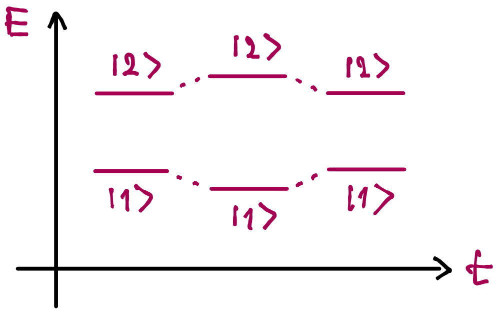
It follows that

$$\rho_{np}\rho_{pn} \le \rho_{nn}\rho_{pp} \Rightarrow \rho = \begin{pmatrix} \rho_{nn} & \cdots & \rho_{np} \\ \vdots & \ddots & \vdots \\ \rho_{pn} & \cdots & \rho_{pp} \end{pmatrix}$$

with = for pure states

Density Matrix Description of 2-Level Atoms

Example: 2-level atom w/random perturbations



Perturbing events cause random phase shifts $e^{i\theta}$ between states.

The ensemble average $\rho_{np} = \langle c_n c_p^* e^{i\theta} \rangle_k$

is reduced by the randomly fluctuating phase

Dipole Radiation:

$$\langle \hat{\vec{p}} \rangle = Tr[\rho \hat{\vec{p}}] = Tr\left[\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\begin{pmatrix} 0 & \vec{p}_{12} \\ \vec{p}_{21} & 0 \end{pmatrix}\right]$$

$$= \rho_{12}\vec{p}_{21} + \rho_{21}\vec{p}_{12} = 2\,Re[\rho_{12}\vec{p}_{21}]$$

For an ensemble of pure states w/different θ

$$\langle \hat{\vec{p}} \rangle = \langle \underbrace{Re[\rho_{12}^{(k)}\vec{p}_{21}]} \rangle_k$$

Oscillating dipole w/phase that varies between atoms with different perturbation history

Time Evolution of the Density Matrix

Challenge: We need "equations of motion" that combine the Schrödinger Equation with the effect of processes that can change $Tr\,\rho^2$ (measure of purity)

Approach: We do not have time for a rigorous derivation, so will rely on plausible arguments to justify the equations

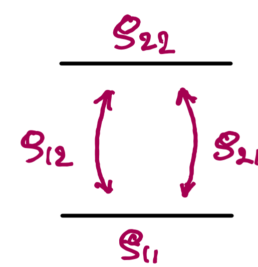
Schrödinger Evolution: In general, we have

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] = -\frac{i}{\hbar}(H\rho - \rho H)$$

matrix elements

$$\dot{\rho}_{nm} = -\frac{i}{\hbar}\sum_{k=1,2}(H_{nk}\rho_{km} - \rho_{nk}H_{km})$$

2-Level Atom ⇨ { 2 populations, 2 coherences }



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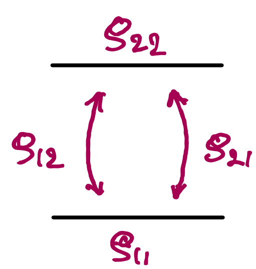
$$\dot{\rho}_{nm} = -\frac{i}{\hbar}\sum_{k=1,2}\left(H_{nk}\rho_{km} - \rho_{nk}H_{km}\right) \quad (*)$$

2-Level Atom ⇨ $\begin{cases} \text{2 populations} \\ \text{2 coherences} \end{cases}$



Consider the 2-Level Rabi problem with

$$H = H_0 + V \quad \& \quad V_{12} = \frac{1}{2}\hbar X_{12} e^{-i\omega t} + c.c.$$

⇩

$$H = \hbar\begin{pmatrix} 0 & \frac{1}{2}\left(X_{12}e^{-i\omega t} + X_{12}^* e^{i\omega t}\right) \\ \frac{1}{2}\left(X_{21}e^{-i\omega t} + X_{21}^* e^{i\omega t}\right) & \Delta \end{pmatrix}$$

Set $X_{12} = X$, $X_{21} = X^*$, substitute $\rho_{12} = \tilde{\rho}_{12} e^{i\omega t}$

slow variable

(Pure state ⇨ $\rho_{12} = a_1 a_2^* = c_1(c_2 e^{-i\omega t})$)

Substitute in $(*)$ (LHS of the page), make RWA, and drop ~ **Homework Set 3 Assignment**

⇩

$$\boxed{\begin{aligned} \dot{\rho}_{11} &= -\frac{i}{2}\left(X\rho_{12} - X^*\rho_{21}\right) \\ \dot{\rho}_{22} &= \frac{i}{2}\left(X\rho_{12} - X^*\rho_{21}\right) \\ \dot{\rho}_{12} &= -i\Delta\rho_{12} + i\frac{X^*}{2}\left(\rho_{22} - \rho_{11}\right) = \dot{\rho}_{21}^* \end{aligned}}$$

Rabi Eqs. for pure and mixed states

Density Matrix Description of 2-Level Atoms

Consider the 2-Level Rabi problem with

$$H = H_0 + V \quad \& \quad V_{12} = \frac{1}{2}\hbar \chi_{12} e^{-i\omega t} + c.c.$$

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Substitute in (∗), make RWA, and drop ~

$$\dot{\rho}_{11} = -\frac{i}{2}(\chi\rho_{12} - \chi^*\rho_{21})$$

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Rabi Eqs. for pure and mixed states

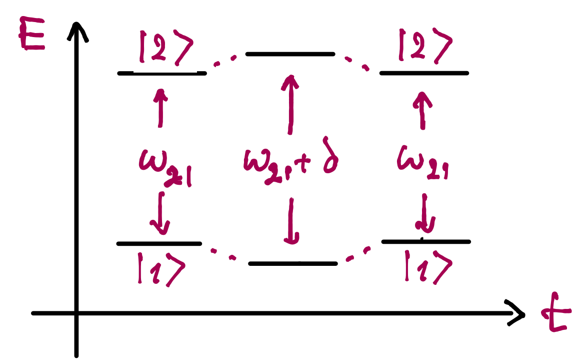
Next: Non-Hamiltonian evolution

Types of events

- **(i)** **Elastic collisions:** No change in energy
- **(ii)** **Inelastic collisions:** Atom loss
- **(iii)** **Spontaneous decay:** Transition $|2\rangle \rightarrow |1\rangle$

Simple Model of Elastic Collisions

Two atoms near each other ⇒ **energy levels shift, free evol. of** ρ_{12} **changed**



(Paradigm for perturbations that do not lead to net change in energy)

Begin 02-15-2024

Density Matrix Description of 2-Level Atoms

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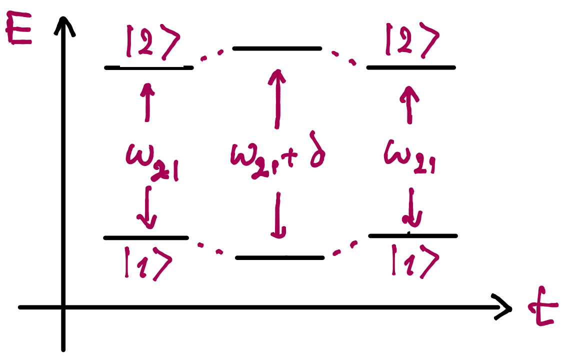

(Paradigm for perturbations that do not lead to net change in energy)

Evolution of coherence (fast variables)

$$\dot{\rho}_{12} = -i\left[\omega_{21} + \delta\omega(t)\right]\rho_{12}$$

collisional history

$$\Rightarrow \rho_{12}(t) = \rho_{12}(0)\, e^{-i\omega_{21}t}\, e^{-i\int_0^t dt'\,\delta\omega(t')}$$

We need the <u>ensemble average of</u> $\rho_{12}(t)$

<u>Assumptions</u>:

- From atom to atom $\delta\omega(t)$ is a Gaussian Random Variable
- Averaged over the ensemble $\langle \delta\omega(t)\rangle_R = 0$
- Collisions have no memory over time,

$$\langle \delta\omega(t)\,\delta\omega(t')\rangle_t = \frac{2}{\tau}\,\delta(t-t')$$

Can show that, averaged over time and the ensemble

$$\left\langle e^{-i\int_0^t dt'\,\delta\omega(t')}\right\rangle_R = e^{-t/\tau}$$

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It follows that: $\rho_{12}(t) = \rho_{12}(0)\, e^{-i\omega_{21}t}\, e^{-t/\tau}$

Add this decay to the equation of motion to get

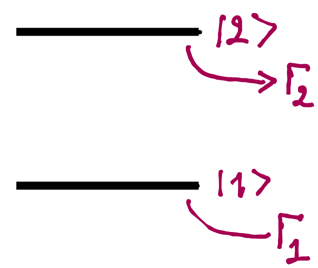
$$\boxed{\dot{\rho}_{12} = (\dot{\rho}_{12})_{S.E.} + (\dot{\rho}_{12})_{E.C.} = -(i\omega_{21} - 1/\tau)\rho_{12}}$$

Simple Model of Inelastic Collisions

As modeled by, e. g., Milloni & Eberly, this is a steady loss of atoms

$$\dot{\rho}_{11} = (\dot{\rho}_{11})_{S.E.} - \Gamma_1\rho_{11}$$

$$\dot{\rho}_{22} = (\dot{\rho}_{22})_{S.E.} - \Gamma_2\rho_{22}$$



This is strange because $Tr\,\rho(t)$ is not preserved

Convenient when working with quantities

$$N\langle\vec{\eta}\rangle \propto N(\vec{\eta}_{12}\rho_{21} + \vec{\eta}_{21}\rho_{12})$$

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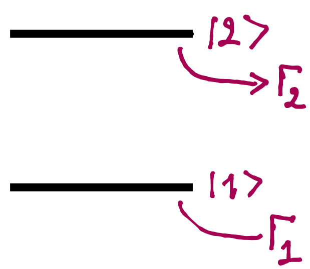

This is strange because $Tr\,\rho(t)$ is not preserved

Convenient when working with quantities

$$N\langle \vec{\mu} \rangle \propto N(\vec{\mu}_{12}\rho_{21} + \vec{\mu}_{21}\rho_{12})$$

Effect on probability amplitudes

Populations are ensemble averages of the type

$$\rho_{11}(t) = \langle |a_1(t)|^2 \rangle = \langle |a_1(0)|^2 \rangle e^{-\Gamma_1 t}$$

$$\rho_{22}(t) = \langle |a_2(t)|^2 \rangle = \langle |a_2(0)|^2 \rangle e^{-\Gamma_2 t}$$

When the populations decay, the averages of the probability amplitudes must decay accordingly,

$$\langle |a_1(t)| \rangle = \langle |a_1(0)| \rangle e^{-\Gamma_1/2\, t}$$

$$\langle |a_2(t)| \rangle = \langle |a_2(t)| \rangle e^{-\Gamma_2/2\, t}$$

Thus, for the coherences

$$\rho_{12}(t) = \langle a_1(t) a_2(t)^* \rangle = \langle a_1(0) a_2(0)^* \rangle e^{-\Gamma_1/2\, t} e^{-\Gamma_2/2\, t}$$

This gives us

$$\boxed{\dot{\rho}_{12} = (\dot{\rho}_{22})_{S.E.} \underbrace{- 1/\tau\, \rho_{12}}_{\text{elastic}} \underbrace{- \frac{\Gamma_1 + \Gamma_2}{2}\rho_{12}}_{\text{inelastic}}}$$

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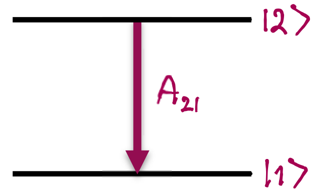
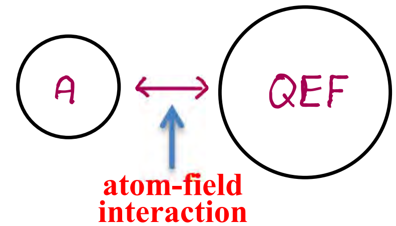
Spontaneous Decay

This process occurs due to interaction between the Atom and the Quantized Electromagnetic Field





Warm-up: A Bayesian recipe for Mixed States

Alice has two 2-level atoms in the ground state.

Step (1) She applies a Hamiltonian that drives the evolution

$$|1\rangle_A |1\rangle_B \rightarrow a_1 |1\rangle_A |1\rangle_B + a_2 |2\rangle_A |2\rangle_B$$

Step (2) She gives atom B to Bob and asks him to measure if it is in $|1\rangle_B$ or $|2\rangle_B$ and keep the result secret forever.

Result: Alice now has a 2-level atom in the state

$$\rho = |a_1|^2 |1\rangle_{BB}\langle 1| + |a_2|^2 |2\rangle_{BB}\langle 2|$$

Density Matrix Description of 2-Level Atoms

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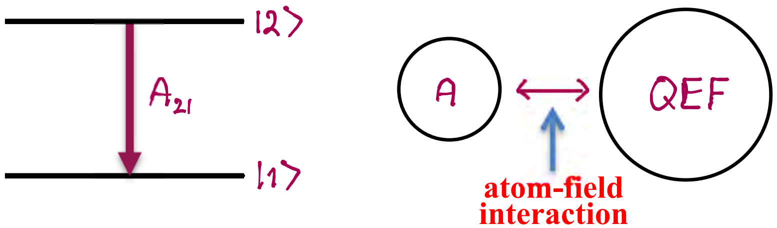

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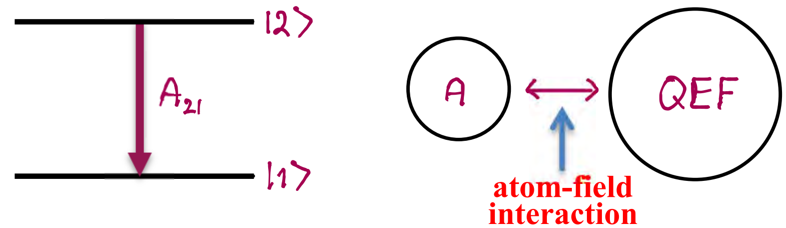


Final OPTI 544 Lectures:

$|\psi(0)\rangle = |2\rangle_A |vac\rangle_{QEF}$ ⇨ evolution over time t

$$|\psi(t)\rangle = C_{2,0}(t)\,|2\rangle_A |vac\rangle_{QEF} ⇨ \sum_k C_{1,1k}(t)\,|1\rangle_A |n_k = 1\rangle_{QEF}$$

photon "in the atom" — photon in field mode k

Cannot recover info in continuum of field modes

Probability $|C_{2,0}(t)|^2$ of having **no decay**

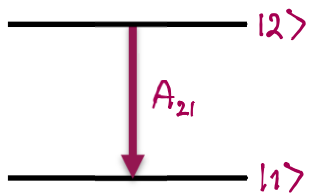
Probability $\sum_k |C_{1,1k}(t)|^2$ of having **decay**

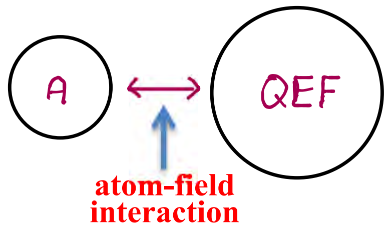
<u>No Coherence</u> established between states $|1\rangle, |2\rangle$

Density Matrix Description of 2-Level Atoms

Spontaneous Decay

This process occurs due to interaction between the Atom and the Quantized Electromagnetic Field





Final OPTI 544 Lectures:

$|\psi(0)\rangle = |2\rangle_A |Vac\rangle_{QEF}$ ➡ evolution over time *t*

$|\psi(t)\rangle = C_{2,0}(t)|2\rangle_A |Vac\rangle_{QEF}$ ➡ $\sum_k C_{1,1k}(t)|1\rangle_A |n_k = 1\rangle_{QEF}$

photon "in the atom" — photon in field mode *k*

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Probability $|C_{2,0}(t)|^2$ of having **no decay**

Probability $\sum_k |C_{1,1k}(t)|^2$ of having **decay**

<u>No Coherence</u> established between states $|1\rangle, |2\rangle$

Conclusion: Decay moves population $|2\rangle \rightarrow |1\rangle$ at rate A_{21}, damps coherence at rate $A_{21}/2$

$$\dot{\rho}_{11} = A_{21}\rho_{22}, \quad \dot{\rho}_{22} = -A_{21}\rho_{11}$$

$$\dot{\rho}_{12} = -\frac{A_{21}}{2}\rho_{12} = \dot{\rho}_{21}^*$$

Putting it all together:

$$\dot{\rho}_{11} = -\Gamma_1 \rho_{11} + A_{21}\rho_{22} - \frac{i}{2}(\chi\rho_{12} - \chi^*\rho_{21})$$

$$\dot{\rho}_{22} = -\Gamma_2 \rho_{22} - A_{21}\rho_{22} + \frac{i}{2}(\chi\rho_{12} - \chi^*\rho_{21})$$

$$\dot{\rho}_{12} = (i\Delta - \beta)\rho_{12} + \frac{i\chi^*}{2}(\rho_{22} - \rho_{11}) = \dot{\rho}_{21}^*$$

where $\beta = \frac{1}{\tau} + \frac{A_{21}}{2} + \frac{\Gamma_1 + \Gamma_2}{2}$

These are our desired **Density Matrix Equations of Motion**

Density Matrix Description of 2-Level Atoms

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More about the Density Matrix

Let $|\psi\rangle = \sum_j c_j |u_j\rangle$, where $\{|u_j\rangle\}$ is a basis and the index k labels members of the ensemble

Populations: $\rho_{nn} = \langle c_n^{(k)} c_n^{(k)*}\rangle_k = \langle |c_j^{(k)}|^2 \rangle_k$

(real-valued)

Bayesianist:
Single system: ρ_{nn} is prob of observing the state $|u_n\rangle$

Frequentist:
Ensemble: The state $|u_n\rangle$ occurs with frequency ρ_{nn}

Coherences: $\rho_{np} = \langle c_n^{(k)} c_p^{(k)*}\rangle_k$, $\rho_{pn} = \rho_{pn}^*$

(complex-valued)

Note: Defining $c_q = |c_q| e^{i\theta_q}$ we have

$$|\langle c_n^{(k)} c_p^{(k)*}\rangle_k| = |\langle |c_n^{(k)}||c_p^{(k)}| e^{i(\theta_n^{(k)} - \theta_p^{(k)})}\rangle_k| \le |\langle |c_n^{(k)}||c_p^{(k)*}|\rangle_k|$$

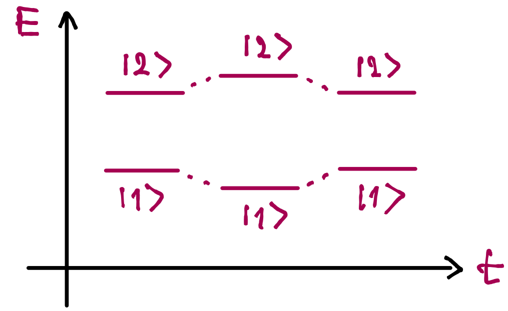
It follows that

$$\rho_{np}\rho_{pn} \le \rho_{nn}\rho_{pp}$$

with = for pure states

$$\rho = \begin{pmatrix} \rho_{nn} & \cdots & \rho_{np} \\ \vdots & \ddots & \vdots \\ \rho_{pn} & \cdots & \rho_{pp} \end{pmatrix}$$

Example: 2-level atom w/random perturbations



Perturbing events cause random phase shifts $e^{i\theta}$ between states.

The ensemble average $\rho_{np} = \langle c_n c_p^* e^{i\theta}\rangle_k$

is reduced by the randomly fluctuating phase

Dipole Radiation:

$$\langle \hat{\vec{p}} \rangle = Tr[\rho\hat{\vec{p}}] = Tr\left[\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\begin{pmatrix} 0 & \vec{p}_{12} \\ \vec{p}_{21} & 0 \end{pmatrix}\right]$$

$$= \rho_{12}\vec{p}_{21} + \rho_{21}\vec{p}_{12} = 2Re[\rho_{12}\vec{p}_{21}]$$

For an ensemble of pure states w/different θ

$$\langle \hat{\vec{p}} \rangle = \langle \underbrace{Re[\rho_{12}^{(k)}\vec{p}_{21}]}\rangle_k$$

Oscillating dipole w/phase that varies between atoms with different perturbation history

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Coherences: (complex-valued) $\rho_{np} = \langle c_n^{(k)} c_p^{(k)*} \rangle_k$, $\rho_{pn} = \rho_{pn}^*$

Note: Defining $c_q = |c_q| e^{i\theta_q}$ we have

$$|\langle c_n^{(k)} c_p^{(k)*} \rangle_k| = |\langle |c_n^{(k)}||c_p^{(k)}| e^{i(\theta_n^{k} - \theta_p^{(k)})} \rangle_k| \leq |\langle |c_n^{(k)}||c_p^{(k)*}| \rangle_k|$$

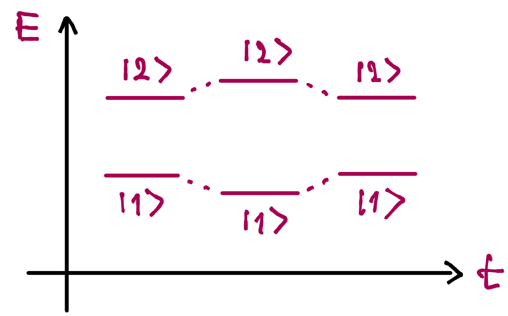
It follows that

$$\rho_{np}\rho_{pn} \leq \rho_{nn}\rho_{pp}$$

with = for pure states

$$\rho = \begin{pmatrix} \rho_{nn} & \cdots & \rho_{np} \\ \vdots & \ddots & \vdots \\ \rho_{pn} & \cdots & \rho_{pp} \end{pmatrix}$$

Example: 2-level atom w/random perturbations



Perturbing events cause random phase shifts $e^{i\theta}$ between states.

The ensemble average $\rho_{np} = \langle c_n c_p^* e^{i\theta} \rangle_k$

is reduced by the randomly fluctuating phase

Dipole Radiation:

$$\langle \hat{\vec{p}} \rangle = Tr[\rho \hat{\vec{p}}] = Tr\left[\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} 0 & \vec{p}_{12} \\ \vec{p}_{21} & 0 \end{pmatrix}\right]$$

$$= \rho_{12}\vec{p}_{21} + \rho_{21}\vec{p}_{12} = 2 Re[\rho_{12}\vec{p}_{21}]$$

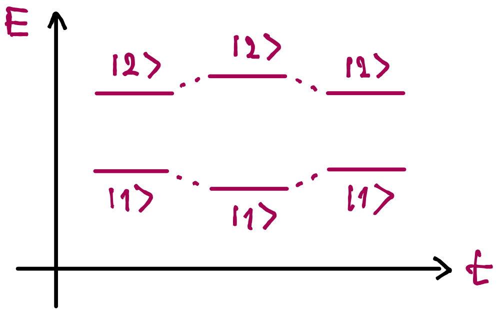
For an ensemble of pure states w/different θ

$$\langle \hat{\vec{p}} \rangle = \langle \underbrace{Re[\rho_{12}^{(k)} \vec{p}_{21}]} \rangle_k$$

Oscillating dipole w/phase that varies between atoms with different perturbation history

Density Matrix Description of 2-Level Atoms

Example: 2-level atom w/random perturbations



Perturbing events cause random phase shifts $e^{i\theta}$ between states.

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is reduced by the randomly fluctuating phase

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Time Evolution of the Density Matrix

Challenge: We need "equations of motion" that combine the Schrödinger Equation with the effect of processes that can change $Tr\, \rho^2$ (measure of purity)

Approach: We do not have time for a rigorous derivation, so will rely on plausible arguments to justify the equations

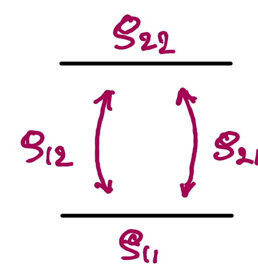
Schrödinger Evolution: In general, we have

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] = -\frac{i}{\hbar}(H\rho - \rho H)$$

matrix elements

$$\dot{\rho}_{nm} = -\frac{i}{\hbar} \sum_{k=1,2} (H_{nk} \rho_{km} - \rho_{nk} H_{km})$$

2-Level Atom ⇨ { 2 populations, 2 coherences }



Density Matrix Description of 2-Level Atoms

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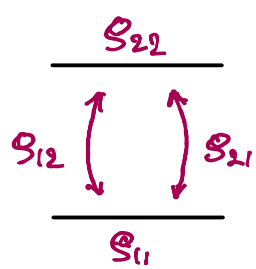
2-Level Atom ⇒ $\begin{cases} \text{2 populations} \\ \text{2 coherences} \end{cases}$



Consider the 2-Level Rabi problem with

$$H = H_0 + V \quad \& \quad V_{12} = \frac{1}{2}\hbar X_{12}e^{-i\omega t} + c.c.$$

$$H = \hbar\begin{pmatrix} 0 & \frac{1}{2}\left(X_{12}e^{-i\omega t} + X_{12}^* e^{i\omega t}\right) \\ \frac{1}{2}\left(X_{21}e^{-i\omega t} + X_{21}^* e^{i\omega t}\right) & \Delta \end{pmatrix}$$

Set $X_{12} = X$, $X_{21} = X^*$, substitute $\rho_{12} = \tilde{\rho}_{12}e^{i\omega t}$

slow variable

(Pure state ⇒ $\rho_{12} = a_1 a_2^* = c_1(c_2 e^{-i\omega t})$)

Substitute in $(*)$ (LHS of the page), make RWA, and drop ~

$$\boxed{\begin{aligned} \dot{\rho}_{11} &= -\frac{i}{2}\left(X\rho_{12} - X^*\rho_{21}\right) \\ \dot{\rho}_{22} &= \frac{i}{2}\left(X\rho_{12} - X^*\rho_{21}\right) \\ \dot{\rho}_{12} &= -i\Delta\rho_{12} + i\frac{X^*}{2}\left(\rho_{22} - \rho_{11}\right) = \dot{\rho}_{21}^* \end{aligned}}$$

Rabi Eqs. for pure and mixed states

Density Matrix Description of 2-Level Atoms

Consider the 2-Level Rabi problem with

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Set $\chi_{12} = \chi$, $\chi_{21} = \chi^*$, **substitute** $\rho_{12} = \tilde{\rho}_{12} e^{i\omega t}$

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(Pure state ⇨ $\rho_{12} = a_1 a_2^* = c_1 (c_2 e^{-i\omega t})$ **)**

Substitute in (*), make RWA, and drop ~

$$\dot{\rho}_{11} = -\frac{i}{2}(\chi \rho_{12} - \chi^* \rho_{21})$$

$$\dot{\rho}_{22} = \frac{i}{2}(\chi \rho_{12} - \chi^* \rho_{21})$$

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Rabi Eqs. for pure and mixed states

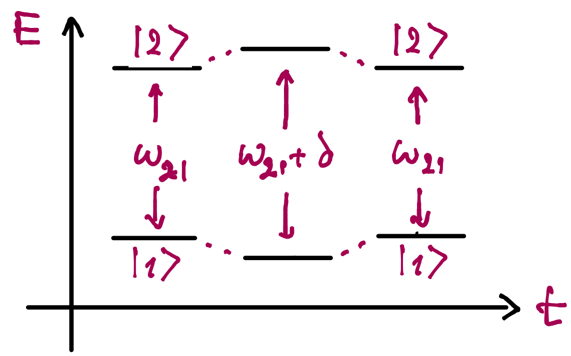
Next: Non-Hamiltonian evolution

Types of events

(i)	**Elastic collisions:**	**No change in energy**
(ii)	**Inelastic collisions:**	**Atom loss**
(iii)	**Spontaneous decay:**	**Transition** $|2\rangle \to |1\rangle$

Simple Model of Elastic Collisions

Two atoms near each other ⇨ **energy levels shift, free evol. of ρ_{12} changed**



(Paradigm for perturbations that do not lead to net change in energy)

Density Matrix Description of 2-Level Atoms

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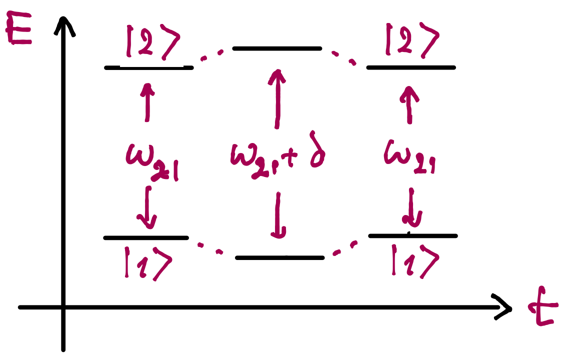

(Paradigm for perturbations that do not lead to net change in energy)

Evolution of coherence (fast variables)

$$\dot{\rho}_{12} = -i\left[\omega_{21} + \delta\omega(t)\right]\rho_{12}$$

collisional history

$$\Rightarrow \rho_{12}(t) = \rho_{12}(0)\, e^{-i\omega_{21}t}\, e^{-i\int_0^t dt'\,\delta\omega(t')}$$

We need the ensemble average of $\rho_{12}(t)$

Assumptions:

- From atom to atom $\delta\omega(t)$ is a Gaussian Random Variable
- Averaged over the ensemble $\langle \delta\omega(t)\rangle_k = 0$
- Collisions have no memory over time,

$$\langle \delta\omega(t)\,\delta\omega(t')\rangle_t = \frac{2}{\tau}\,\delta(t-t')$$

Can show that, averaged over time and the ensemble

$$\left\langle e^{-i\int_0^t dt'\,\delta\omega(t')}\right\rangle_k = e^{-t/\tau}$$

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Density Matrix Description of 2-Level Atoms

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It follows that: $\rho_{12}(t) = \rho_{12}(0)\, e^{-i\omega_{21}t}\, e^{-t/\tau}$

Add this decay to the equation of motion to get

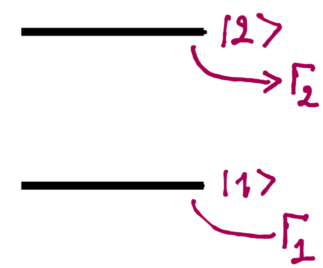
$$\boxed{\dot{\rho}_{12} = (\dot{\rho}_{12})_{S.E.} + (\dot{\rho}_{12})_{E.C.} = -(i\omega_{21} - 1/\tau)\rho_{12}}$$

Simple Model of Inelastic Collisions

As modeled by, e. g., Milloni & Eberly, this is a steady loss of atoms

$$\dot{\rho}_{11} = (\dot{\rho}_{11})_{S.E.} - \Gamma_1\rho_{11}$$

$$\dot{\rho}_{22} = (\dot{\rho}_{22})_{S.E.} - \Gamma_2\rho_{22}$$



This is strange because $Tr\,\rho(t)$ is not preserved

Convenient when working with quantities

$$N\langle\vec{\eta}\rangle \propto N(\vec{\eta}_{12}\rho_{21} + \vec{\eta}_{21}\rho_{12})$$

Density Matrix Description of 2-Level Atoms

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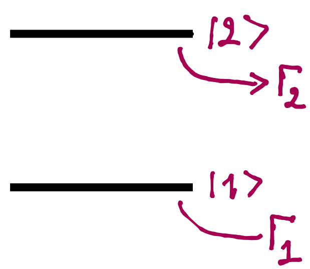

This is strange because $Tr\, \rho(t)$ is not preserved

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$$N\langle \vec{\mu} \rangle \propto N(\vec{\mu}_{12}\rho_{21} + \vec{\mu}_{21}\rho_{12})$$

Effect on probability amplitudes

Populations are ensemble averages of the type

$$\rho_{11}(t) = \langle |a_1(t)|^2 \rangle = \langle |a_1(0)|^2 \rangle e^{-\Gamma_1 t}$$
$$\rho_{22}(t) = \langle |a_2(t)|^2 \rangle = \langle |a_2(0)|^2 \rangle e^{-\Gamma_2 t}$$

When the populations decay, the averages of the probability amplitudes must decay accordingly,

$$\langle |a_1(t)| \rangle = \langle |a_1(0)| \rangle e^{-\Gamma_1/2\, t}$$
$$\langle |a_2(t)| \rangle = \langle |a_2(t)| \rangle e^{-\Gamma_2/2\, t}$$

Thus, for the coherences

$$\rho_{12}(t) = \langle a_1(t) a_2(t)^* \rangle = \langle a_1(0) a_2(0)^* \rangle e^{-\Gamma_1/2\, t} e^{-\Gamma_2/2\, t}$$

This gives us

$$\boxed{\dot{\rho}_{12} = (\dot{\rho}_{22})_{S.E.} \underbrace{- 1/\tau\, \rho_{12}}_{\text{elastic}} \underbrace{- \frac{\Gamma_1 + \Gamma_2}{2} \rho_{12}}_{\text{inelastic}}}$$

Density Matrix Description of 2-Level Atoms

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elastic inelastic

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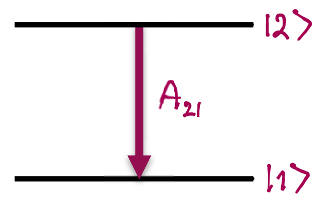
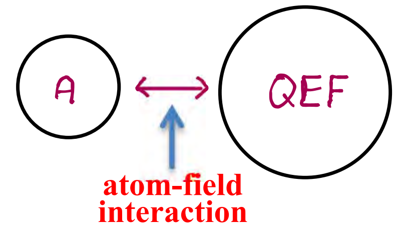
Spontaneous Decay

This process occurs due to interaction between the Atom and the Quantized Electromagnetic Field





Warm-up: A Bayesian recipe for Mixed States

Alice has two 2-level atoms in the ground state.

Step (1) She applies a Hamiltonian that drives the evolution

$$|1\rangle_A |1\rangle_B \rightarrow a_1 |1\rangle_A |1\rangle_B + a_2 |2\rangle_A |2\rangle_B$$

Step (2) She gives atom B to Bob and asks him to measure if it is in $|1\rangle_B$ or $|2\rangle_B$ and keep the result secret forever.

Result: Alice now has a 2-level atom in the state

$$\rho = |a_1|^2 |1\rangle_{BB}\langle 1| + |a_2|^2 |2\rangle_{BB}\langle 2|$$

Density Matrix Description of 2-Level Atoms

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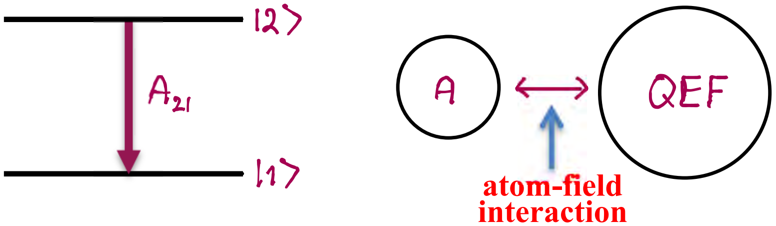

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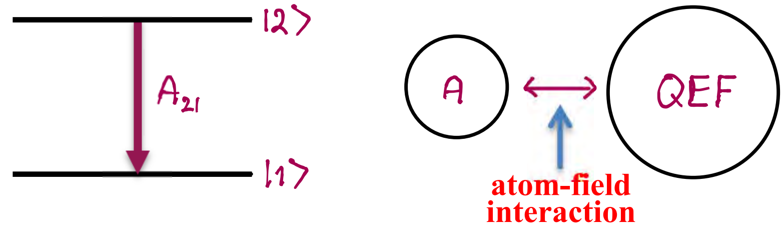

Final OPTI 544 Lectures:

$|\psi(0)\rangle = |2\rangle_A |vac\rangle_{QEF}$ ➡ evolution over time t

$$|\psi(t)\rangle = C_{2,0}(t)|2\rangle_A|vac\rangle_{QEF} \;➡\; \sum_k C_{1,1k}(t)|1\rangle_A|n_k=1\rangle_{QEF}$$

photon "in the atom" photon in field mode k

Cannot recover info in continuum of field modes

Probability $|C_{2,0}(t)|^2$ of having no decay

Probability $\sum_k |C_{1,1k}(t)|^2$ of having decay

<u>No Coherence</u> established between states $|1\rangle, |2\rangle$

Density Matrix Description of 2-Level Atoms

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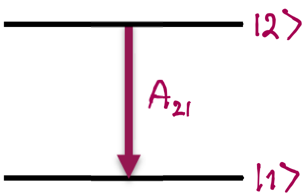

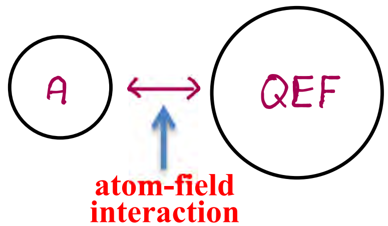

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Conclusion: Decay moves population $|2\rangle \to |1\rangle$ **at rate** A_{21}**, damps coherence at rate** $A_{21}/2$

$$\dot{\rho}_{11} = A_{21}\rho_{22}, \quad \dot{\rho}_{22} = -A_{21}\rho_{11}$$

$$\dot{\rho}_{12} = -\frac{A_{21}}{2}\rho_{12} = \dot{\rho}_{21}^*$$

Putting it all together:

$$\dot{\rho}_{11} = -\Gamma_1\rho_{11} + A_{21}\rho_{22} - \frac{i}{2}(\chi\rho_{12} - \chi^*\rho_{21})$$

$$\dot{\rho}_{22} = -\Gamma_2\rho_{22} - A_{21}\rho_{22} + \frac{i}{2}(\chi\rho_{12} - \chi^*\rho_{21})$$

$$\dot{\rho}_{12} = (i\Delta - \beta)\rho_{12} + \frac{i\chi^*}{2}(\rho_{22} - \rho_{11}) = \dot{\rho}_{21}^*$$

where $\beta = \frac{1}{\tau} + \frac{A_{21}}{2} + \frac{\Gamma_1 + \Gamma_2}{2}$

These are our desired **Density Matrix Equations of Motion**