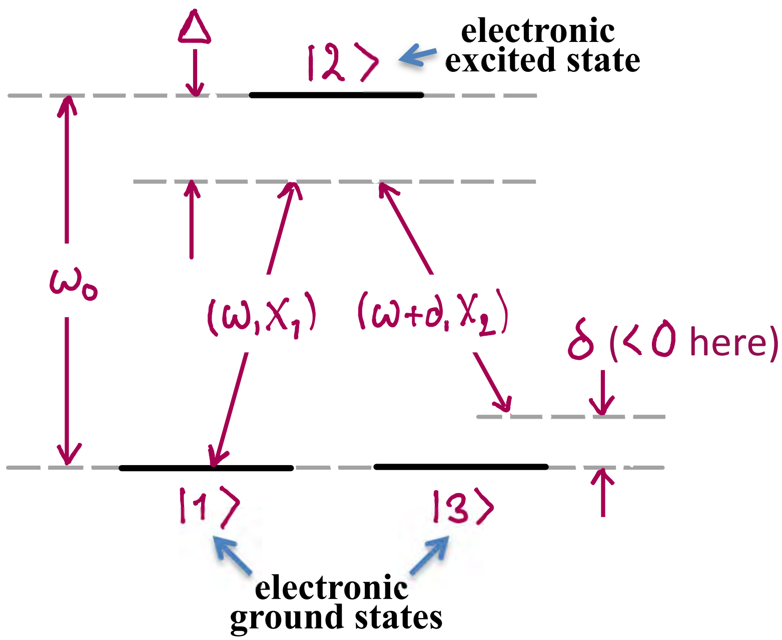
Raman Coupling in 3-level Atoms

Raman Coupling in 3-Level Systems

Consider an atom with this 3-level structure



For simplicity we set $E_1 = E_3$ (no loss of generality)

Fields $\begin{cases} \text{at } \omega, \text{ coupling } |1\rangle, |2\rangle \text{ w/Rabi freq. } \chi_1 \\ \text{at } \omega+\delta, \text{ coupling } |3\rangle, |2\rangle \text{ w/Rabi freq. } \chi_2 \end{cases}$

The Hamiltonian for this system is (χ_1, χ_2 real)

$$\boxed{H = \hbar \begin{pmatrix} 0 & \chi_1(t) & 0 \\ \chi_1(t) & \omega_0 & \chi_2(t) \\ 0 & \chi_2(t) & 0 \end{pmatrix}}$$

where

$$\chi_1(t) = \frac{\chi_1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$\chi_2(t) = \frac{\chi_2}{2}\left(e^{i(\omega+\delta)t} + e^{-i(\omega+\delta)t}\right)$$

Setting $|\psi(t)\rangle = a_1(t)|1\rangle + a_2(t)|2\rangle + a_3(t)|3\rangle$ **we get a S.E.**

$$\dot{a}_1 = -i\frac{\chi_1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)a_2$$

$$\dot{a}_2 = -i\omega_0 a_2 - i\frac{\chi_1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)a_1 - i\frac{\chi_2}{2}\left(e^{i(\omega+\delta)t} + e^{-i(\omega+\delta)t}\right)a_3$$

$$\dot{a}_3 = -i\frac{\chi_2}{2}\left(e^{i(\omega+\delta)t} + e^{-i(\omega+\delta)t}\right)a_2$$

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Raman Coupling in 3-level Atoms

(5) Finally, the last term $\propto \frac{e^{-i\Delta t}}{\Delta}$ can be ignored because it averages to zero on the timescale on which b_1, b_3 evolve.

Note:

The ground state amplitudes evolve slowly Because $\chi_1/\Delta, \chi_2/\Delta \ll 1$, while the excited state amplitude evolves fast and adiabatically follows the instantaneous values of b_1, b_3

Plug the solution for $b_2(t)$ into the eqs. for b_1, b_3

$$\boxed{\begin{aligned} \dot{b}_1(t) &= i\frac{\chi_1^2}{4\Delta} b_1(t) + i\frac{\chi_1\chi_2}{4\Delta} b_3(t) \\ \dot{b}_3(t) &= -i\left(\delta - \frac{\chi_2^2}{4\Delta}\right) b_3(t) + i\frac{\chi_1\chi_2}{4\Delta} b_1(t) \end{aligned}}$$

We simplify by making a final change of variables

$$c_1(t) = b_1(t)\, e^{-i\frac{\chi_1^2}{4\Delta}t}, \quad c_3(t) = b_3(t)\, e^{-i\frac{\chi_1^2}{4\Delta}t}$$

$$\boxed{\begin{aligned} \dot{c}_1(t) &= i\frac{\chi_1\chi_2}{4\Delta} c_3(t) \\ \dot{c}_3(t) &= -i\left(\delta + \frac{\chi_1^2 - \chi_2^2}{4\Delta}\right) c_3(t) + i\frac{\chi_1\chi_2}{4\Delta} c_1(t) \end{aligned}}$$

These are two-level equations!

Physical Discussion: We have an effective 2-level atom with effective Rabi Frequency and detuning.

$$\boxed{\chi_{eff} = \frac{\chi_1\chi_2}{2\Delta}, \quad \delta_{eff} = \delta + \frac{\chi_1^2 - \chi_2^2}{4\Delta}}$$

Note that $\chi_{eff} \sim \chi^2/\Delta$ while the excited state population $P_2 \sim \chi^2/\Delta^2$. This means that for large χ, Δ we can have large χ_{eff} and no opportunity for spontaneous decay.

Coherent Rabi oscillations and long lived superposition states

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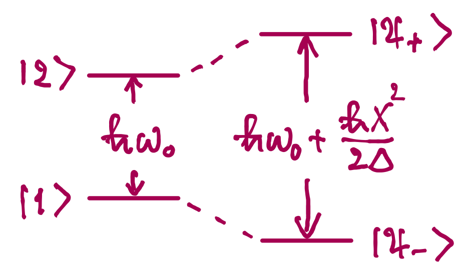
Coherent Rabi oscillations and long lived superposition states

Note also: The effective Raman detuning is shifted.

HW Set 2: Dressed-states of a 2-level atom



$$E_+ = \hbar\omega_0 + \frac{\hbar\chi^2}{4\Delta}$$

$$E_- = -\frac{\hbar\chi^2}{4\Delta}$$

3-level system ⇨ ground state shifts $\frac{\chi_1^2}{4\Delta}, \frac{\chi_2^2}{4\Delta}$

⇨ Differential ground state shift $\frac{\chi_1^2-\chi_2^2}{4\Delta}$

Final note: The atomic dipole $\langle \vec{\wp} \rangle$ **will have components that match the frequency and polarization of both driving fields, with amplitudes that depend on both fields.**

Non-Linear wave mixing,
Breakdown of superposition principle

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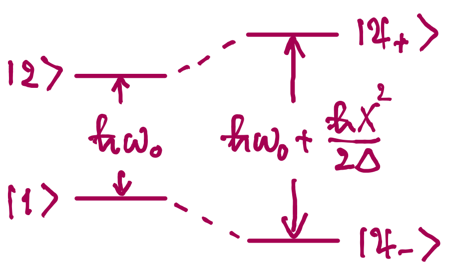

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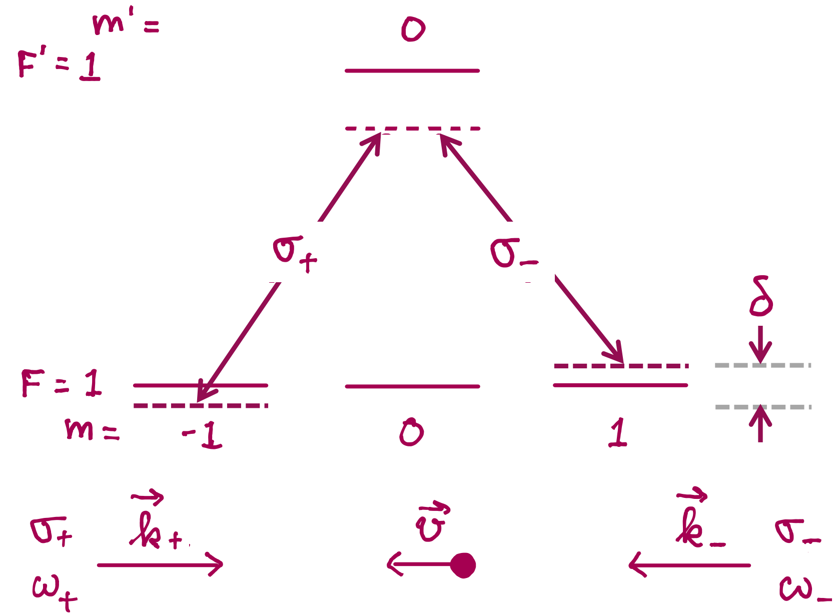
Example: Velocity dependent Raman Coupling



field freqs. in co-moving frame

$$\left.\begin{aligned} \omega_+ &= \omega + kv \\ \omega_- &= \omega - kv \end{aligned}\right\}$$

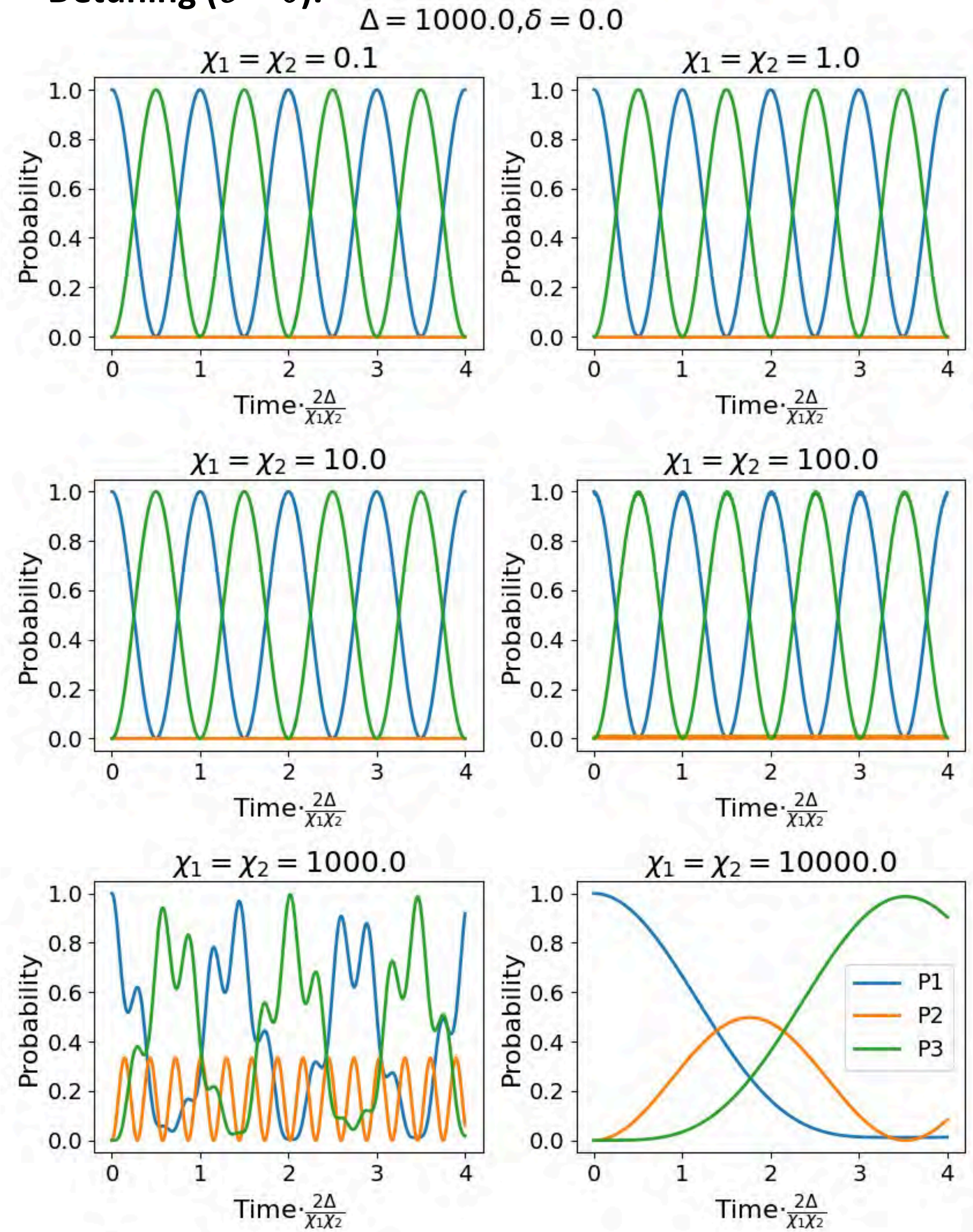
velocity dependent Raman detuning

➡ $\delta = 2kv$

Applications:

- Doppler velocimetry
- Raman Cooling by velocity selective momentum transfer
- What if we apply a $\pi/2$ Raman pulse?
- Atom Interferometry

Numerical integration of the equations for the probability amplitudes in a 3-level Lambda system with zero Raman Detuning ($\delta = 0$).



Density Matrix Description of 2-Level Atoms

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Mental Warmup: What is a probability?

(1) Example: Coin toss

- We can describe physical states by probability distributions
- Probabilities are assigned based on prior knowledge, updated when additional info becomes available
- As such, probability distributions are subjective (states of knowledge)

(2) Example: Quincunx

https://www.mathsisfun.com/data/quincunx.html

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This is the Bayesian Interpretation of Probability

(3) Example: Quantum Quincunx

- We can describe physical states by quantum wavefunctions (state vectors)
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(4) Mixed Quantum & Classical Case

- We can easily envision a hybrid Quincunx that is part classical, part quantum.
- Physics needs an efficient description these kinds of intermediate situations

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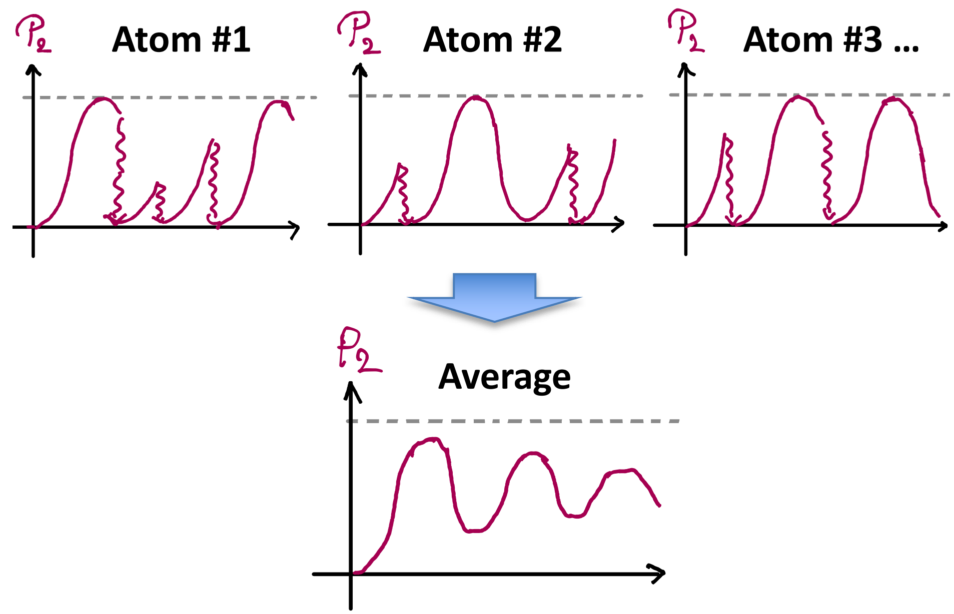
- Ensemble of 2-level atoms undergoing Rabi oscillation with random decays



Definition: A system for which we know only the probabilities p_k of finding the system in state $|\psi_k\rangle$ is said to be in a statistical mixture of states. Shorthand: mixed state.

Shorthand for non-mixed state: pure state

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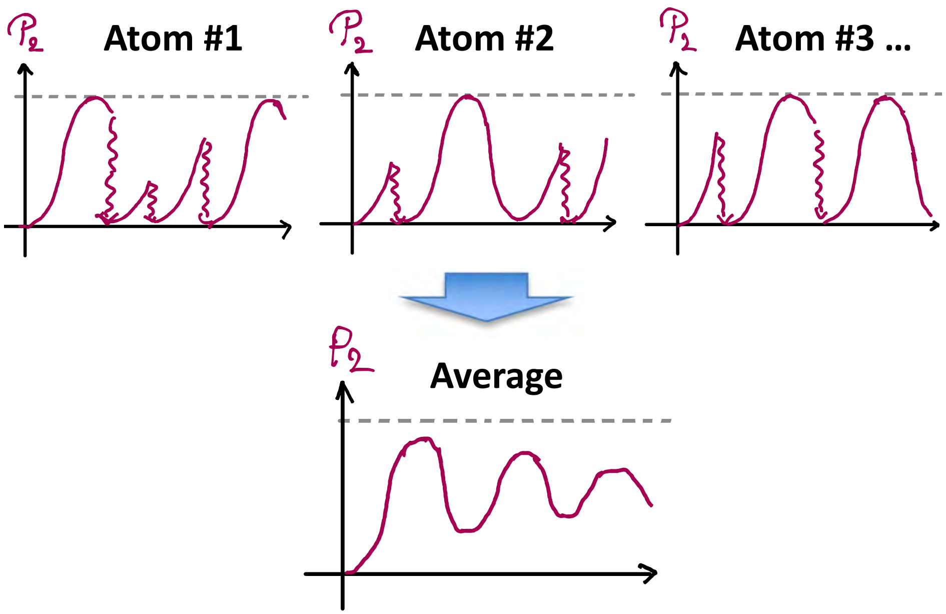

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Definition: Density Operator for pure states

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)|$$

Definition: Density Matrix

$$|\psi(t)\rangle = \sum_n c_n(t)|u_n\rangle \Rightarrow$$

$$\rho_{pn}(t) = \langle u_p|\rho(t)|u_n\rangle = c_p(t)\,c_n^*(t)$$

Definition: Density Operator for mixed states

$$\rho(t) = \sum_{\vec{k}} p_k\,\rho_k(t)\,, \quad \rho_k = |\psi_k(t)\rangle\langle\psi_k(t)|$$

Note: A pure state is just a mixed state for which one $p_k = 1$ and the rest are zero.

The terms Density Operator and Density Matrix are used interchangeably

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Density Matrix Description of 2-Level Atoms

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The terms Density Operator and Density Matrix are used interchangeably

Let A be an observable w/eigenvalues a_n

Let P_n be the projector on the eigen-subspace of a_n

For a pure state, $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$, we have

(∗) $\mathrm{Tr}\,\rho(t) = \sum_n \rho_{nn}(t) = \sum_n |C_n|^2 = 1$

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$= \mathrm{Tr}[\rho(t)A]$ ($|u_p\rangle$ basis in $\mathcal{H}$)

(∗) Let P_n be the projector on eigensubspace of a_n

$$\mathcal{P}(a_n) = \langle\psi(t)|P_n|\psi(t)\rangle = \mathrm{Tr}[\rho(t)P_n]$$

(∗) $\dot{\rho}(t) = |\dot{\psi}(t)\rangle\langle\psi(t)| + |\psi(t)\rangle\langle\dot{\psi}(t)|$

$= \frac{1}{i\hbar} H|\psi(t)\rangle\langle\psi(t)| - \frac{1}{i\hbar}|\psi(t)\rangle\langle\psi(t)|H$

$= \frac{1}{i\hbar}[H,\rho]$

Density Matrix Description of 2-Level Atoms

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Density Operator formalism is general !

Density Matrix Description of 2-Level Atoms

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Density Operator formalism is general !

Important properties of the Density Operator

(1) ρ is Hermitian, $\rho^\dagger = \rho$ ➡ ρ is an observable

➡ $\exists$ basis in which ρ is diagonal

In this basis a pure state has <u>one</u> diagonal element $= 1$, the rest $= 0$

(2) Test for purity.

Pure: $\rho^2 = \rho$ ➡ $\mathrm{Tr}\,\rho^2 = 1$

Mixed: $\rho^2 \neq \rho$ ➡ $\mathrm{Tr}\,\rho^2 < 1$

(3) Schrödinger evolution does not change the p_k

➡ $\left\{\begin{array}{l}\mathrm{Tr}\,\rho^2 \text{ is conserved} \\ \text{pure states stay pure} \\ \text{mixed states stay mixed}\end{array}\right.$

Changing pure ➡ mixed requires non-Hamiltonian evolution – see Cohen Tannoudji D_{III} & E_{III}

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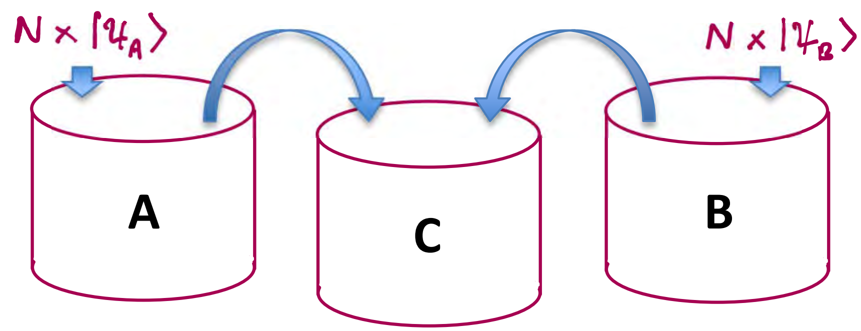
A cooks recipe – interpretations of ρ

Step 1 Add ***N*** atoms in state $|\psi_A\rangle$ to bucket A

Add ***N*** atoms in state $|\psi_B\rangle$ to bucket B

We now have two ensembles, each of which consist of ***N*** atoms in a known pure state

Step 2 Add buckets A and B to bucket C and stir.



Pick an atom from C

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$\rho = \frac{1}{2}|\psi_A\rangle\langle\psi_A| + \frac{1}{2}|\psi_B\rangle\langle\psi_B|$

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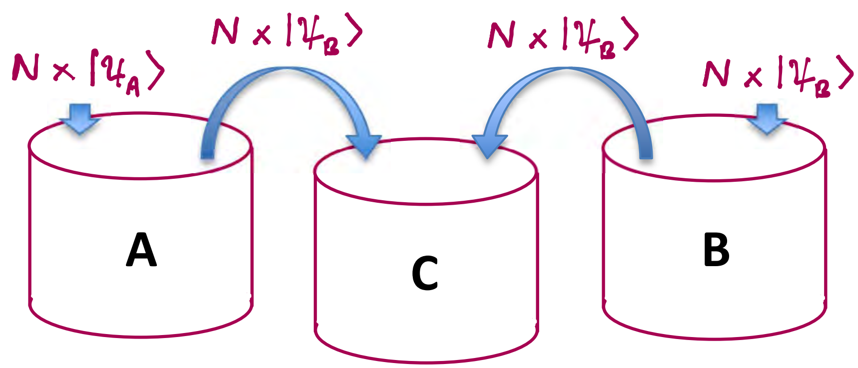

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There is no difference!

The two interpretations lead to identical predictions for any measurement we can do on atoms from C

Quantum Mechanics:

If two descriptions lead to identical predictions for observable outcomes then they are identical

Loosely, (i) is a *frequentist view*

(ii) is a *Bayesian view*

Quantum Bayesianism

Quantum States are States of Knowledge

(subjective)

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Density Matrix Description of 2-Level Atoms

More about the Density Matrix

Let $|\psi\rangle = \sum_j c_j |u_j\rangle$, where $\{|u_j\rangle\}$ is a basis and the index k labels members of the ensemble

Populations: $\rho_{nn} = \langle c_n^{(k)} c_n^{(k)*} \rangle_k = \langle |c_j^{(k)}|^2 \rangle_k$
(real-valued)

Bayesianist:
Single system: ρ_{nn} is prob of observing the state $|u_n\rangle$

Frequentist:
Ensemble: The state $|u_n\rangle$ occurs with frequency ρ_{nn}

Coherences: $\rho_{np} = \langle c_n^{(k)} c_p^{(k)*} \rangle_k$, $\rho_{pn} = \rho_{pn}^*$
(complex-valued)

Note: Defining $c_q = |c_q| e^{i\theta_q}$ we have

$$|\langle c_n^{(k)} c_p^{(k)*} \rangle_k| = |\langle |c_n^{(k)}||c_p^{(k)}| e^{i(\theta_n^{k} - \theta_p^{(k)})} \rangle_k| \leq |\langle |c_n^{(k)}||c_p^{(k)*}| \rangle_k|$$

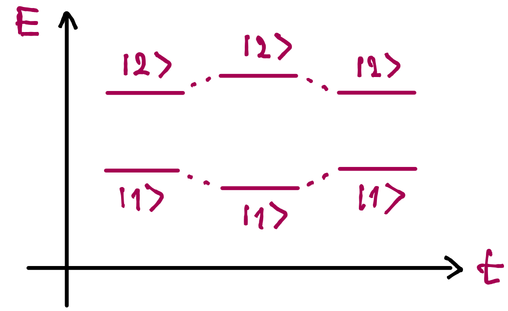
It follows that

$$\rho_{np}\rho_{pn} \leq \rho_{nn}\rho_{pp}$$

with = for pure states

$$\rho = \begin{pmatrix} \rho_{nn} & \cdots & \rho_{np} \\ \vdots & \ddots & \vdots \\ \rho_{pn} & \cdots & \rho_{pp} \end{pmatrix}$$

Example: 2-level atom w/random perturbations



Perturbing events cause random phase shifts $e^{i\theta}$ between states.

The ensemble average $\rho_{np} = \langle c_n c_p^* e^{i\theta} \rangle_k$

is reduced by the randomly fluctuating phase

Dipole Radiation:

$$\langle \hat{\vec{p}} \rangle = Tr[\rho \hat{\vec{p}}] = Tr\left[\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\begin{pmatrix} 0 & \vec{p}_{12} \\ \vec{p}_{21} & 0 \end{pmatrix}\right]$$

$$= \rho_{12}\vec{p}_{21} + \rho_{21}\vec{p}_{12} = 2 Re[\rho_{12}\vec{p}_{21}]$$

For an ensemble of pure states w/different θ

$$\langle \hat{\vec{p}} \rangle = \langle \underbrace{Re[\rho_{12}^{(k)} \vec{p}_{21}]} \rangle_k$$

Oscillating dipole w/phase that varies between atoms with different perturbation history

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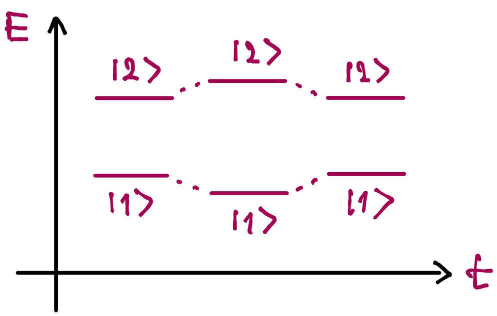
Density Matrix Description of 2-Level Atoms

Example: 2-level atom w/random perturbations



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Time Evolution of the Density Matrix

Challenge: We need "equations of motion" that combine the Schrödinger Equation with the effect of processes that can change $Tr\,\rho^2$ (measure of purity)

Approach: We do not have time for a rigorous derivation, so will rely on plausible arguments to justify the equations

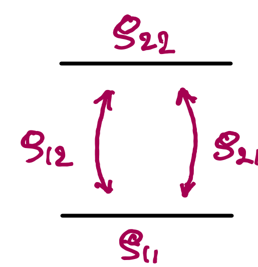
Schrödinger Evolution: In general, we have

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] = -\frac{i}{\hbar}(H\rho - \rho H)$$

matrix elements

$$\dot{\rho}_{nm} = -\frac{i}{\hbar}\sum_{k=1,2}(H_{nk}\rho_{km} - \rho_{nk}H_{km})$$

2-Level Atom ⇨ { 2 populations, 2 coherences



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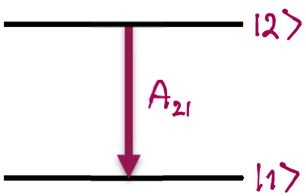
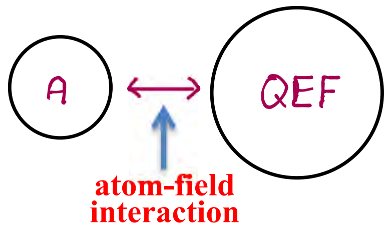
Density Matrix Description of 2-Level Atoms

Spontaneous Decay

This process occurs due to interaction between the Atom and the Quantized Electromagnetic Field





Final OPTI 544 Lectures:

$|\psi(0)\rangle = |2\rangle_A |vac\rangle_{QEF}$ ➡ evolution over time *t*

$|\psi(t)\rangle = C_{2,0}(t)\,|2\rangle_A |vac\rangle_{QEF}$ ➡ $\sum_k C_{1,1k}(t)\,|1\rangle_A |n_k = 1\rangle_{QEF}$

photon "in the atom" photon in field mode *k*

Cannot recover info in continuum of field modes

Probability $|C_{2,0}(t)|^2$ of having no decay

Probability $\sum_k |C_{1,1k}(t)|^2$ of having decay

No Coherence established between states $|1\rangle, |2\rangle$

Conclusion: Decay moves population $|2\rangle \rightarrow |1\rangle$ at rate A_{21}, damps coherence at rate $A_{21}/2$

$$\dot{\rho}_{11} = A_{21}\rho_{22}\ , \quad \dot{\rho}_{22} = -A_{21}\rho_{11}$$

$$\dot{\rho}_{12} = -\frac{A_{21}}{2}\rho_{12} = \dot{\rho}_{21}^{*}$$

Putting it all together:

$$\dot{\rho}_{11} = -\Gamma_1\rho_{11} + A_{21}\rho_{22} - \frac{i}{2}\left(\chi\rho_{12} - \chi^*\rho_{21}\right)$$

$$\dot{\rho}_{22} = -\Gamma_2\rho_{22} - A_{21}\rho_{22} + \frac{i}{2}\left(\chi\rho_{12} - \chi^*\rho_{21}\right)$$

$$\dot{\rho}_{12} = (i\Delta - \beta)\rho_{12} + \frac{i\chi^*}{2}\left(\rho_{22} - \rho_{11}\right) = \dot{\rho}_{21}^*$$

where $\beta = \frac{1}{\tau} + \frac{A_{21}}{2} + \frac{\Gamma_1 + \Gamma_2}{2}$

These are our desired **Density Matrix Equations of Motion**

Emission and Absorption – Population Rate Equations