Quantum Theory of Light-Matter Interaction

Completed:

– Fully classical description of fields & Atoms

Next Step:

– Semiclassical description $\left\{\begin{array}{l}\text{Classical field}\\ \text{Quantum atoms}\end{array}\right.$

Self-Consistent Description

Electromagnetic Field → Atom/Molecule/Solid

Needed: Quantum theory of atomic response

analogous to classical $\vec{p} = \alpha(\omega)\,\vec{E}$

Note: In QM the dipole is an Observable

Observable = Hermitian operator $\hat{\vec{p}}$

Classical Field = C-valued vector $\vec{E}$

Cannot plug into Wave Eq. for classical field !

Wave Equation w/classical field & atoms

$$\boxed{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\vec{P}, \qquad \vec{P} = N\vec{p}}$$

How do we deal with this mis-match?

Repeated measurements of $\vec{p}(t)$

⇓

Quantum fluctuations $\vec{p}(t) = \langle\vec{p}(t)\rangle + \Delta\vec{p}(t)$

where $\langle\vec{p}(t)\rangle = \langle\psi(t)|\hat{\vec{p}}\,|\psi(t)\rangle$ mean

$\Delta\hat{\vec{p}}(t)$ fluctuations

Note: Given $|\psi(t=0)\rangle$ and $\vec{E}$ the mean $\langle\vec{p}(t)\rangle$

follows from the Schrödinger Eq.,

radiates <u>coherently</u> like classical $\vec{p}(t)$

$\langle\vec{p}(t)\rangle$ is a Real-valued vector (more later)

⇨ we can plug it into the Wave Eq.

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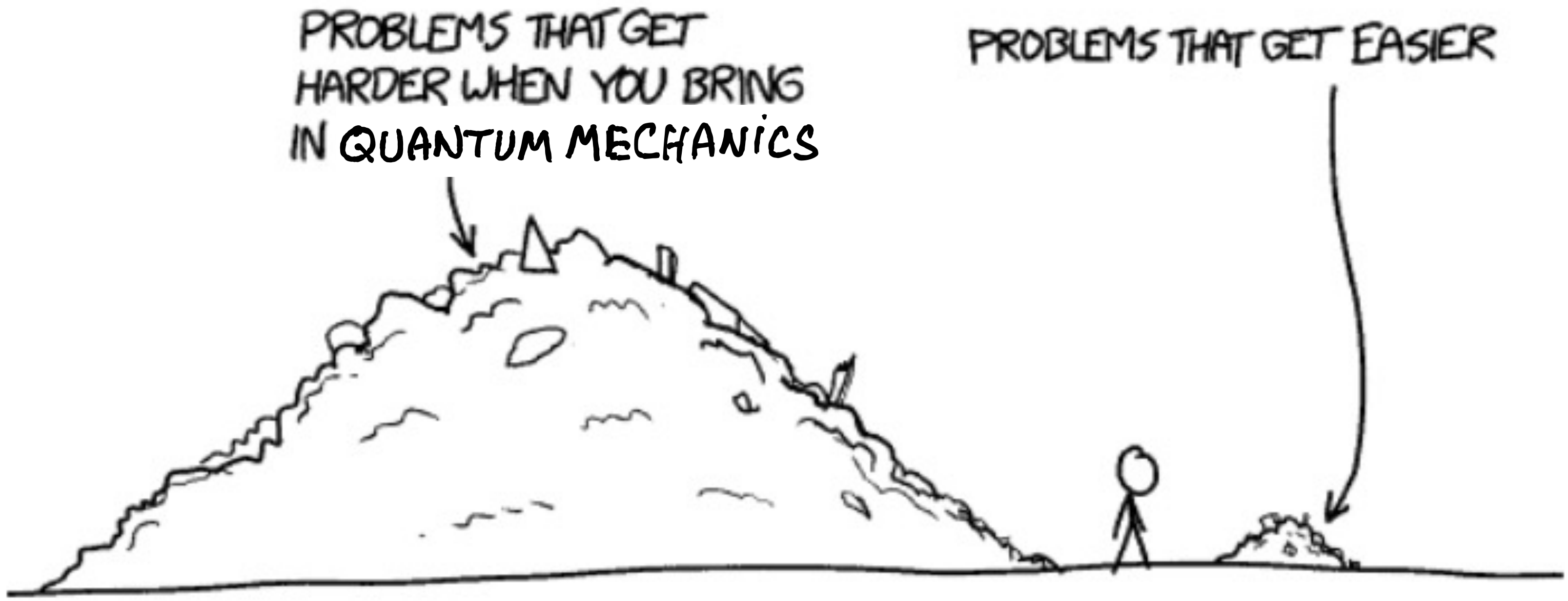

Source: xkcd.com

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Wave Equation w/classical field & atoms

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Classical Wave EQ w/Quantum atoms

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\vec{P}, \qquad \vec{P} = N\langle\vec{p}\rangle$$

Note:
- The Equations look very similar
- Polarizability, index of refraction, etc will be *very different* in some regimes
- Notably, the model is no longer linear in $\vec{E}$ and will lead to phenomena like saturation and wave mixing
- $\Delta\hat{\vec{p}}(t)$ represents quantum fluctuations driven by the empty modes of the EM field, a process also responsible for spontaneous decay.

Note: Do not identify $\langle\vec{p}\rangle$ and $\Delta\vec{p}$ with Stimulated and spontaneous emission. Those labels are not meaningful here.

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Wave Equation w/classical field & atoms

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Wave Eq. w/classical field & quantum atoms

$$\boxed{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\vec{P}, \quad \vec{P} = N\langle\vec{\mu}\rangle}$$

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Atom-field interaction

Hamiltonian: $\boxed{H = H_a + V_{ext}(\vec{R},t)}$

H_a : time-independent atomic Hamiltonian

V_{ext} : time-dependent driving term, non necessarily a perturbation

Question: Time evolution of the atomic system? Is there a steady state?

Schrödinger Eq.: $\boxed{i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle}$

Expand in basis $\{|\varphi_n\rangle\}$ of eigenstates of H_a

$$|\psi(t)\rangle = \sum_n a_n(t)|\varphi_n\rangle, \quad H_a|\varphi_n\rangle = E_n|\varphi_n\rangle$$

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Plug into S. E.

$$i\hbar\sum_n \dot{a}_n(t)|\varphi_n\rangle = \sum_n a_n(t)\left[E_n + V_{ext}\right]|\varphi_n\rangle$$

Take scalar product w/ $|\varphi_m\rangle$ on both sides

$$i\hbar\sum_n \dot{a}_n(t)\langle\varphi_m|\varphi_n\rangle = \sum_n a_n(t)\left[E_n\langle\varphi_m|\varphi_n\rangle\right] + \overbrace{\langle\varphi_m|V_{ext}|\varphi_n\rangle}^{V_{mn}}$$

On vector-matrix form this can be written

$$\boxed{i\hbar\underline{\dot{a}} = \underline{\underline{H}}_a\,\underline{a} + \underline{\underline{V}}\,\underline{a}}$$

S.E. in $\{|\varphi_n\rangle\}$ rep.

Still exact!