Light-Matter Interaction

Classical Theory of Light-Matter Interaction

Self-consistent, fully classical description

Electromagnetic Field $\rightarrow$ Atom/Molecule/Solid

Motivation: We will

- Develop Concepts $\alpha(\omega)$, n, χ
- Develop Intuition
- Classical is often adequate, sometimes accurate
- A Quantum Theory has classical limits ⇨ Identify/understand regime of validity
- The Classical description is a useful starting point for Nonlinear Optics

The Electromagnetic Field: Basic Eqs. in SI Units

Maxwell's eqs.

(no free charges, currents ⇨ dielectrics)

(i) $\nabla \cdot \vec{D} = \rho = 0$ — $\vec{D}$: Dielectric displacement

(ii) $\nabla \cdot \vec{B} = 0$ — $\vec{B}$: Magnetic induction

(iii) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ — $\vec{E}$: Electric field

(iv) $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$ — $\vec{H}$: Magnetic field

Material Response:

(v) $\vec{B} = \mu_0 \vec{H} + \vec{M}$ ⇦ Non-magnetic ⇨ $\vec{M} = 0$

(vi) $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ ⇦ Info about response in dipole moment density (polarization density)

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We need equations that describe:

- the behavior of $\vec{E}$ for given $\vec{P}$
- the medium response $\vec{P}$ for given $\vec{E}$

Wave Equation:

Take curl of (iii), then use (iv)

$$\nabla\times(\nabla\times\vec{E}) = -\nabla\times\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla\times\vec{B}) = -\mu_0\frac{\partial^2\vec{D}}{\partial t^2}$$

Next, use the identity

$$\nabla\times(\nabla\times\vec{E}) = \nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E}$$

to obtain $\vec{D} = \nabla(\nabla\cdot\vec{E}) = -\mu_0\frac{\partial^2\vec{D}}{\partial t^2}$

Finally, let $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$ and use $\varepsilon_0\mu_0 = \frac{1}{c^2}$

to obtain

$$\boxed{-\nabla(\nabla\cdot\vec{E}) + \nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} + \frac{1}{\varepsilon_0 c^2}\frac{\partial^2\vec{P}}{\partial t^2}}$$

This is the Wave Equation, still exact in this form

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Transverse Fields

Definition: a field for which $\nabla\cdot\vec{E} \equiv 0$ is Transverse

Example: a plane wave, $\vec{E}(\vec{r},t) = \vec{E}(t)\,e^{i\vec{k}\cdot\vec{r}}$,

where $\vec{E}(t)\perp\vec{k}$, is transverse.

The physical field is $Re[\vec{E}(\vec{r},t)]$

For transverse fields the wave equation simplifies to

$$\boxed{\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\vec{P}}$$

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Isotropic Media

Absent a preferred direction, the induced $\vec{P}$ must be parallel to the driving field $\vec{E}$

Linear response, most general case:

$$\vec{D}(t) = \varepsilon_0 \vec{E}(t) + \vec{P}(t)$$
$$= \varepsilon_0 \vec{E}(t) + \varepsilon_0 \int_{-\infty}^{t} dt' R(t-t')\vec{E}(t')$$

where the *response function* $R(t-t')$ is a scalar and we have $R(\tau) = 0$ for $\tau < 0$

Take $\nabla \cdot$ on both sides, divide by ε_0 & use M.E. (i)

$$\nabla \cdot \vec{D}(t) = \varepsilon_0 \nabla \cdot \vec{E}(t) + \varepsilon_0 \int_{-\infty}^{t} dt' R(t-t')\nabla \cdot \vec{E}(t') = 0$$

$$\Rightarrow \nabla \cdot \vec{E}(t) = -\int_{-\infty}^{t} dt' R(t-t')\nabla \cdot \vec{E}(t') \quad \text{for all } t$$

It follows that $\nabla \cdot \vec{E}(t) = 0$ (Transverse) for all t

<u>OR</u> $R(\tau) = -2\delta(\tau)$

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OR $R(\tau) = -2\delta(\tau)$

Note: if $R(\tau) \propto \delta(\tau)$ (instantaneous response) then

$$\varepsilon_0 \int_{-\infty}^{t} dt' R(t-t') \vec{E}(t') = \varepsilon_0 \chi \vec{E}(t)$$

χ — susceptibility

The case $R(\tau) = -2\delta(\tau)$ is an example of negative susceptibility, $\chi < 0$, which only occurs in certain engineered metamaterials.

$\Downarrow$

Electric fields are transverse in linear, isotropic dielectric media

(including the vacuum)

Wave Equation in free space

$$\boxed{\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = 0}$$

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Monochromatic trial solution $\vec{E}(\vec{r},t) = \vec{E}_0(\vec{r})\, e^{-i\omega t}$

$$\nabla^2 \vec{E}_0(\vec{r}) e^{-i\omega t} + \frac{\omega^2}{c^2}\vec{E}_0(\vec{r}) e^{-i\omega t} = 0$$

Equation for the spatial component alone:

$$\nabla^2 \vec{E}_0(\vec{r}) + |\vec{k}|^2 \vec{E}_0(\vec{r}) = 0 \, , \; |\vec{k}| = \omega/c$$

Plane wave solutions

$$\boxed{\vec{E}_0(\vec{r}) = \vec{\varepsilon} E_0 e^{i\vec{k}\cdot\vec{r}} , \; |\vec{k}| = \omega/c}$$

<u>Optical Cavities</u>: Here we need to solve the wave equation subject to boundary conditions. See, e. g., Millony & Eberly for examples such as rectangular cavities, Fabryt-Perot etalons, and spherical mirror resonators.

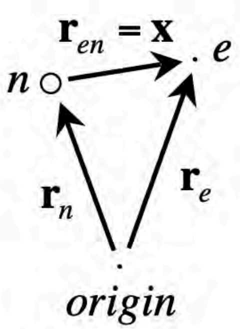
Light-Matter Interaction

Theory of Atomic Response

So far, we have a model for the field. Next, we need a model of how the constituents of the medium responds to the field.

This will allow us to find the polarization density $\vec{P}$ as function of the field $\vec{E}$

Classical "atom"



Simple model:
nucleus + electron

Lorentz Force

$$\boxed{\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})}$$

~ 0 if non-relativistic

Newton:

$$\text{(i)} \quad m_n \frac{d^2}{dt^2}\vec{r}_n(t) = -e\vec{E}(\vec{r}_n, t) - \vec{F}_{en}(\vec{r}_{en}, t)$$

$$\text{(ii)} \quad m_e \frac{d^2}{dt^2}\vec{r}_e(t) = e\vec{E}(\vec{r}_e, t) + \vec{F}_{en}(\vec{r}_{en}, t)$$

This is a standard 2-body problem which we can re-cast as in terms of relative and COM motion.

We define:

$$\vec{x} = \vec{r}_{en} = \vec{r}_e - \vec{r}_n \qquad m = \frac{m_e m_n}{m_e + m_n} \sim m_e$$

$$\vec{R} = \frac{m_e \vec{r}_e + m_n \vec{r}_n}{M} \qquad M = m_e + m_n \sim m_n$$

$\vec{x}$ Relative coord. $\quad m$ Reduced mass

$\vec{R}$ Center-of-mass $\quad M$ Total mass

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Sub into (i), (ii) and rewrite:

$$M\frac{d^2}{dt^2}\vec{R} = e\left[\vec{E}\left(\vec{R} + \frac{m_n}{M}\vec{x},t\right) - \vec{E}\left(\vec{R} - \frac{m_e}{M}\vec{x},t\right)\right]$$

$$m\frac{d^2}{dt^2}\vec{x} = \frac{e}{2}\left[\vec{E}\left(\vec{R} + \frac{m_n}{M}\vec{x},t\right) + \vec{E}\left(\vec{R} - \frac{m_e}{M}\vec{x},t\right)\right] + \vec{F}_{en}(\vec{x}) + \frac{1}{2}(m_n - m_e)\frac{d^2}{dt^2}\vec{R}$$

Basic result, no approximations !

Milloni & Eberly, main text ⇨ **Set** $\vec{R} \approx \vec{r}_n$, $\vec{x} \approx \vec{r}_{en}$

Throw away eq. for $\vec{R}$

<u>Electric Dipole approximation</u>

Atomic dimensions		**Optical Wavelength**
$\lvert\vec{x}\rvert \sim 1\text{Å}$	$\ll$	$\lambda \sim 10^4\text{Å}$

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EDA: the field is nearly constant on the scale of an atom

Good approximation: 1st order expansion in $\vec{x}$

$$\vec{E}\left(\vec{R}-\frac{m_e}{M}\vec{x},t\right) \approx \vec{E}(\vec{R},t) - \frac{m_e}{M}(\vec{x}\cdot\nabla)\vec{E}(\vec{R}(t))$$

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Physical Interpretation:

$\vec{p}=e\vec{x}$: **<u>electric dipole moment of the atom</u>**

The Eqs. Of motion can then be recast as

$$M\frac{d^2}{dt^2}\vec{R}\approx(\vec{p}\cdot\nabla)\vec{E}(\vec{R},t)=\vec{F}=-\nabla_R V(\vec{x},\vec{R},t)$$

$$m\frac{d^2}{dt^2}\vec{x}=e\vec{E}(\vec{R},t)+\vec{F}_{en}(\vec{x})=-\nabla_x V(\vec{x},\vec{R},t)$$

where $V(\vec{x},\vec{r},t)=-\vec{p}\cdot\vec{E}(\vec{r},t)$

<u>electric-dipole interaction</u>

<u>Note</u>: The COM Eq. is the foundation for a range of laser Atom Traps and Optical Tweezers. We will not explore this further in OPTI 544 lectures, but good review articles can be found in the published literature.

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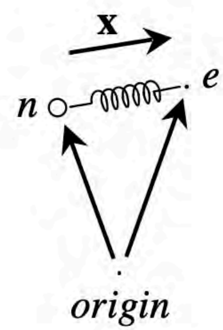
The Electron Oscillator/Lorentz Oscillator

Simple model w/a harmonically bound electron:



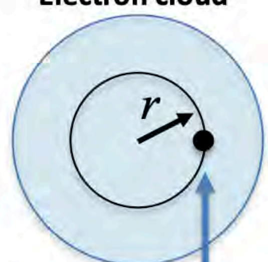
$\vec{F}_{en} = -m\omega_0^2\vec{x}$

resonance frequency

This is meant as a model of the atomic <u>response</u>, not a model of the atom itself.

<u>Nevertheless</u>: QM suggest the atom consists of a point-like nucleus and a spherical electron cloud

Electron cloud



nucleus

Force from charge inside r as if entire charge was at the center

Force from charge outside r is zero

Force $F \propto \frac{r^3}{r^2} \propto r$

harmonic restoring force

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Now substitute $\vec{F}_{en} = -m\omega_0^2 \vec{x}$ into eq. for $\vec{x}$

$$\boxed{\frac{d^2}{dt^2}\vec{x} + \omega_0^2 \vec{x} = \frac{e}{m}\vec{E}(\vec{R},t)}$$

Combine with $\vec{P} = N\vec{\mu}$, $\vec{\mu} = e\vec{x}$ where N is the number density of atoms. This relates the macroscopic $\vec{P}$ to the microscopic $\vec{x}$

We now have

Maxwell's Equations The Lorentz model

Maxwell-Lorentz Equations
We can seek self-consistent solutions to wave propagation

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Classical Model of Absorption & Dispersion