Final Exam 2024 - Problem Set

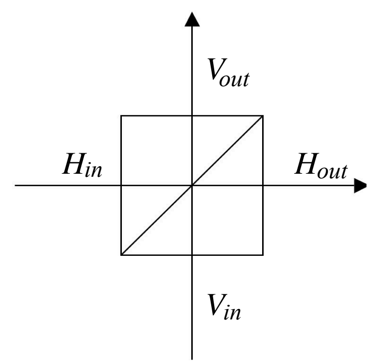
Problem I

What is a photon? Is it a particle, a wave, or something else? What do we mean when we invoke wave-particle duality and how can we observe it? Discuss in your own words. Extra points for clarity. (20%)

Problem II

We typically think of the quantum beam splitter as applying a discrete input-output map between incoming and outgoing modes. If we confine ourselves to single-photon states, then the inputs and outputs are two-level systems with orthogonal basis states

$$|1_H, 0_V\rangle_{in}, |0_H, 1_V\rangle_{in} \quad \text{and} \quad |1_H, 0_V\rangle_{out}, |0_H, 1_V\rangle_{out}$$



(a) Let $t = 1/\sqrt{2}$, $r = i/\sqrt{2}$. Given a two-mode input $|\Psi\rangle_{in} = |1_H, 0_V\rangle_{in}$, find the two-mode output $|\Psi\rangle_{out}$. (10%)

Clearly, for two input modes and exactly one photon, we are dealing with a two-level system for which all possible states can be expressed in the orthonormal basis $|1_H, 0_V\rangle$, $|0_H, 1_V\rangle$. That means all possible states of the two-modes-and-a-photon system can be visualized on the Bloch sphere. Adapt the convention $|1\rangle = |1_H, 0_V\rangle$ and $|2\rangle = |0_H, 1_V\rangle$ on both input and output sides.

(b) Draw a Bloch sphere showing the location of the input and output states from (a). From this, comment on the nature of the transformation that the beam splitter implements on the Bloch sphere. Intuitively, how do you think the position of $|\Psi\rangle_{out}$ will change as the transmission t goes from $t = 1$ to $t = 0$ and the reflection ir goes from $r = 0$ to $r = 1$? (10%)

(c) Would the above representation of states on the Bloch sphere be possible if we had more than one photon present in the two-mode system? Explain in your own words. (10%)

Problem III

Consider in the following a Rubidium atom confined to a pair of Rydberg states that can be considered stable against decay on the time scales relevant here. That makes it effectively a two-level atom with a transition frequency of $\nu = 51\,\text{GHz}$ and a dipole moment $|\vec{p}_{12}| = 1.6 \times 10^{-29}\,\text{C m}$. It is located at the antinode of a 1D standing wave inside a cylindrical cavity of length $\lambda/2$, designed so the mode polarization is parallel to $\vec{p}_{12}$ and the *effective* mode volume is λ^3. The cavity field frequency is tuned to be exactly resonant with the atomic transition frequency.

(a) To what temperature must we cool the cavity in order to keep the mean photon number in the mode below 10^{-3}? Your final answer must be a temperature in degrees K. (15%)

(b) Assume the atom is in the upper Rydberg state and the cavity mode in the vacuum state at $t = 0$. Calculate the frequency of oscillation between the Rydberg states. Your final answer must be a number with units. (15%)

(c) What is this phenomenon called? What might prevent us from observing it in a real experiment? Given the large dipole moment, why can we ignore spontaneous decay from the upper to the lower Rydberg level? Hint: Consider the expression for A_{21} derived in the final lecture of the semester (20%)

Final Exam 2024 - Solution Set

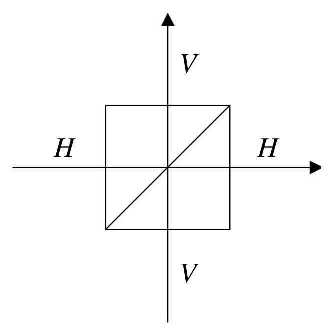
Problem I

(a) It depends! A formal answer would be that a photon is a quantum of excitation in a normal mode of the electromagnetic field. In that case we might elaborate a bit further and say it depends on the nature of the normal mode in which the photon is created, and the kind of measurements we do at later times. To elaborate still further, we can reasonably define the key particle-like behavior as indivisibility, and the key wave-like property as interference. In that case the quintessential demonstration of wave-particle duality is Wheeler's Delayed Choice experiment. It features a Mach-Zender interferometer in which the input beam splitter is fixed, and the output beam splitter can be inserted or removed at random after the photon has entered the interferometer through the first beam splitter. The presence or absence of the second beam splitter then determines whether we observe wavelike behavior (interference between two paths), or particle-like behavior (always one single photon emerging in one or the other output port.) Causality means that nothing happening at the input beam splitter can influence what happens at the output, so we are forced to accept that the photon will behave like a particle if we look for particle-like behavior, and a wave if we look for wavelike behavior.

Problem II

Input and Output states are coupled two-level systems with orthogonal basis states

$$|1_H, 0_V\rangle_{in}, |0_H, 1_V\rangle_{in} \quad \text{and} \quad |1_H, 0_V\rangle_{out}, |0_H, 1_V\rangle_{out}$$



(a) Assume $t = 1/\sqrt{2}$, $r = i/\sqrt{2}$ and the input state is $|\Psi\rangle_{in} = |1_H, 0_V\rangle_{in}$.

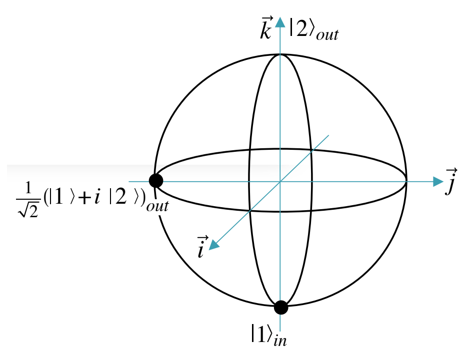
Using $\hat{a}_H^+ \to (t\hat{a}_H^+ + r\hat{a}_V^+)$ we easily find that

$$|\Psi\rangle_{in} = \hat{a}_H^\dagger |0_H, 0_V\rangle_{in} \to |\Psi\rangle_{out} = \left(\frac{1}{\sqrt{2}}\hat{a}_H^\dagger + \frac{i}{\sqrt{2}}\hat{a}_V^\dagger\right)|0_H, 0_V\rangle_{out}$$

$$= \frac{1}{\sqrt{2}}(|1_H, 0_V\rangle_{out} + \frac{i}{\sqrt{2}}|0_H, 1_V\rangle)_{out} = \frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle)_{out}$$

where we have adapted the convention $|1\rangle = |1_H, 0_V\rangle$, $|2\rangle = |0_H, 1_V\rangle$ on both input and output sides.

(b) Position of the $|\Psi\rangle_{in}$ and $|\Psi\rangle_{out}$ states on the Bloch sphere. Refering to the notes on the Optical Bloch picture we recognize the particular superposition of $(|1\rangle + i|2\rangle)/\sqrt{2}$ as a Bloch vector pointing in the $-j$ direction. This shows that our 50-50 beam splitter implements the equivalent of a $\pi/2$ rotation around the i axis.



It is easy to see that a beam splitter with $t = 0$, $r = -i$ does not couple the two two-level systems at all. We know that $t = 1/\sqrt{2}$, $r = i/\sqrt{2}$ lands the output as shown on the right, and it is easy to see that $t = 0$, $r = i$ will land the output at the top of the Bloch sphere. This suggests we can access the full continuum of rotations in the *j*-*k* plane by choosing *t*, *r* appropriately.

(c) If we have more than one photon present, the number of input states would be > 2, and the same for the output states. In that case the input and output states can no longer be visualized on the Block sphere.

Problem III

(a) The mean photon number in the mode is

$$\bar{n} = \frac{q}{1-q} = 10^{-3} \;\Rightarrow\; q = \frac{\bar{n}}{1+\bar{n}} \approx 10^{-3} = e^{-\hbar\omega_{21}/k_B T}.$$ Solving for T gives us

$$T \;=\; \hbar\omega_{21} / \ln(10^3) k_B \;=\; \frac{1.05\times 10^{-34}\,\mathrm{Js}\times 2\pi\times 51\times 10^{9}\,\mathrm{s}^{-1}}{6.91\times 1.381\times 10^{-23}\,\mathrm{J/K}} \;=\; \underline{\underline{0.35\mathrm{K}}}$$

(b) This is the QED version of the classic Rabi problem, and the frequency of the Rabi oscillation between the two states is

$$2|g| = \frac{2|\vec{p}_{ij}|}{\hbar}\sqrt{\frac{\hbar\omega_{12}}{2\varepsilon_0 V}} = |\vec{p}_{ij}|\sqrt{\frac{2\omega_{12}}{\hbar\varepsilon_0 (c/\nu)^3}}$$

$$= 1.6\times 10^{-29}\,\mathrm{C\,m}\sqrt{\frac{4\pi\times 51\times 10^{9}\,\mathrm{s}^{-1}}{1.05\times 10^{-34}\,\mathrm{Js}\times 8.854\times 10^{-12}\,\mathrm{F/m}\times[(3\times 10^{8}\,\mathrm{m/s})/(51\times 10^{9}\,Hz)]^3}}$$

$$= 931\,\mathrm{s}^{-1} \approx 148\,\mathrm{Hz}$$

(c) This phenomenon (the fact that the atom evolves out of the excited state even though there are no photons in the field) is referred to as vacuum Rabi oscillation. To make it observable the lifetimes of the atomic states and the photon lifetime in the cavity must all be $\geq\; (2|g|)^{-1} \sim 20$ ns. So if those lifetimes are too short, we might not see it. In this particular version of the experiment, we might also be prevented from seeing vacuum Rabi oscillations if the cavity temperature is too high and there are thermal photons present. As for why spontaneous decay is unimportant, the basic intuition is that $\mathrm{A}_{21} \propto \omega^3$, a factor that is reduced by $\sim 10^{13}$ when going from an optical to a microwave frequency. We can be more quantitative by explicitly calculating the spontaneous decay rate for the two-level system,

$$A_{21} = \frac{(2\pi\nu)^3|\vec{p}_{21}|^2}{3\varepsilon_0\hbar\lambda^3} = 3.6\times 10^{-5}\,\mathrm{s}^{-1} \quad\Rightarrow\quad \frac{2|g|}{A_{21}} \;=\; 2.6\times 10^{7} \;>>\; 1$$

This shows spontaneous decay is completely negligible on the timescale of the vacuum Rabi oscillation. The actual experiment was done by Haroche and co-workers many years ago.———————————