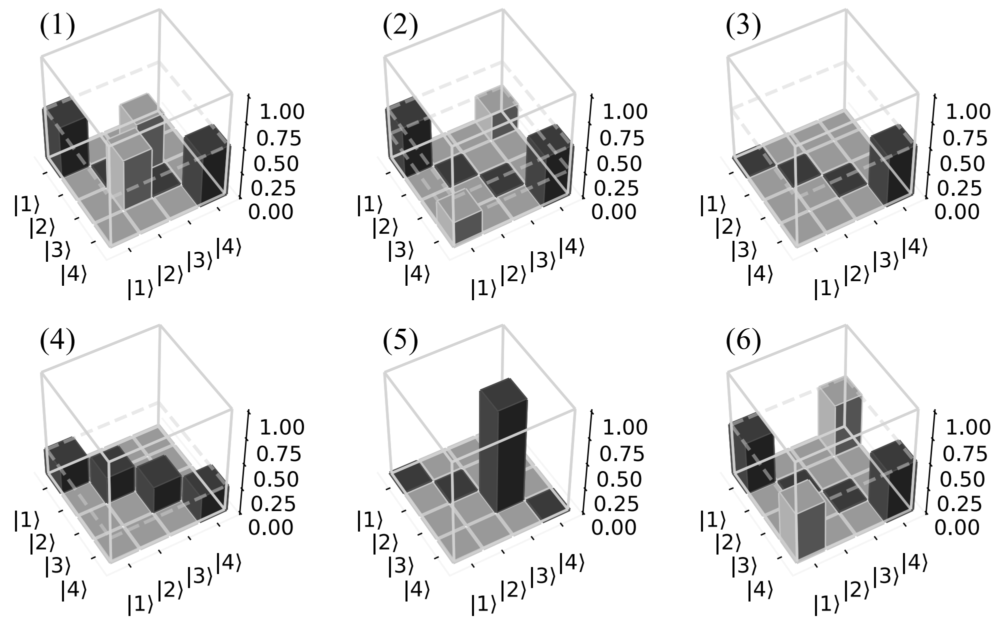
OPTI 544 2nd Midterm 2024 - Problem Set

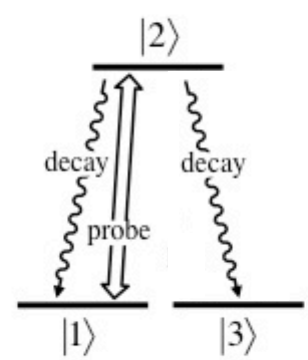
Problem I



The above figure shows a number of "density matrices" for a 4-level system. The dark grey elements along the diagonal are populations; the light gray off-diagonal elements are coherences. The heights of the columns indicate the absolute values of the corresponding populations and coherences.

(a) Some of the density matrices correspond to pure states, some to mixed states, and some are unphysical. Indicate which are which and explain why. Hint: The assignments can be made by visual inspection. (20%)

Problem II



Consider an optical medium consisting of atoms with the level structure shown at right. Spontaneous decay from the excited state into each ground state is equally likely, with decays $|2\rangle \rightarrow |1\rangle$ and $|2\rangle \rightarrow |3\rangle$ happening at rates A_{21} each. The transitions are at wavelength $\lambda = 1.0\,\mu m$ and have associated photon scattering cross sections identical to a similar two-level atom. The medium is a slab of length $l = 2cm$ with atom number density $N = 10^{15} m^{-3}$.

We now send an optical probe beam through the medium. It is resonant with the $|1\rangle \rightarrow |2\rangle$ transition and the photon flux Φ is well below saturation, $\sigma\Phi << A_{21}$. At $t=0$ all atoms are in state $|1\rangle$, $\rho_{11}(t=0)=1$. There is no field present to drive the $|3\rangle \rightarrow |2\rangle$ transition.

(a) Calculate the transmission T at time $t=0$. Your final answer must be a number. (10%)

(b) Find the steady state transmission T_{SS}. Explain. (10%)

Exam continues on back page

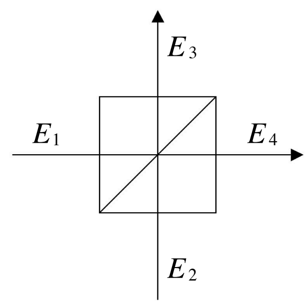
Problem III

(a) Explain in your own words why the photon scattering cross section ends up being a simple function of the transition wavelength. (10%)

(b) What is power broadening, what is its significance when doing spectroscopy, and how can we manage or avoid it. Explain in your own words. (10%)

Problem IV

Consider in the following a 50/50 beam splitter with input ports 1, 2 and output ports 3, 4. Find the output states and the probability of coincidence detection (simultaneous detection of one or more photons in each of the ports 3, 4) for each of the following 2-photon input states.



(a) $|\Psi_{in}\rangle = |1\rangle_1 |1\rangle_2$ (10%)

$|\Psi_{in}\rangle = |2\rangle_1 |0\rangle_2$ (10%)

$|\Psi_{in}\rangle = \frac{1}{\sqrt{2}}\left(|2\rangle_1 |0\rangle_2 + |0\rangle_1 |2\rangle_2\right)$ (20%)

2nd Midterm 2024 - Solution Set

Problem I

(a) (1) By inspection $\rho_{32} \times \rho_{23} > \rho_{22} \times \rho_{33} \Rightarrow$ **Unphysical**

(2) $\rho_{41} \times \rho_{14} < \rho_{11} \times \rho_{44} \Rightarrow$ **Mixed**

(3) $Tr(\rho) < 1 \Rightarrow$ **Unphysical**

(4) All 4 populations equal $\Rightarrow$ (Maximally) **mixed**.

(5) One population = 1, rest = 0 $\Rightarrow$ **Pure**

(6) $\rho_{41} \times \rho_{14} = \rho_{11} \times \rho_{44} \Rightarrow$ **Pure**

Problem II

(a) Initially all atoms are in state $|1\rangle$, and the absorption coefficient for the probe is

$$a_0 = N\sigma = N\frac{3\lambda^2}{2\pi} = 10^{15}m^{-3}\frac{3\times\left(10^{-6}m\right)^2}{2\pi} = 477.47m^{-1}$$

The transmission is then $T_0 = \exp(-a_0 l) = \exp\left(-477.47m^{-1} \times 0.02m\right) = 7.125 \times 10^{-5}$

(b) In steady state all atoms are optically pumped into state $|3\rangle$. With no atoms in state $|1\rangle$ the medium becomes transparent to the probe, $T_{SS} = 1$.

Problem III

(a) The electric dipole approximation rests of the fact that the size of an atom is 4 orders of magnitude less than a visible optical wavelength. Nevertheless, atoms and photons behave as if the atom presents a cross section of order wavelength squared. We can understand this intuitively if we switch the role of the atom and the photon, i. e., we think of the photon as the target and the atom as the impacting particle. But how do we create a stationary, particle-like photon? We can borrow the concept of a normal mode from QED, and recall that as the normal mode becomes more and more localized a photon in that mode becomes more and more particle-like. Because of diffraction the ultimate limit is a particle-like photon confined to a volume of order wavelength cubed. And in that case the particle-like photon presents a cross section of order wavelength squared to the atom. Hence the observed scaling.

(b) Power broadening is a phenomenon that occurs when we do laser spectroscopy to measure the frequency of transitions in atomic media. What happens is that for high probe intensities the ability of the atoms to scatter light will saturate, and this will happen sooner and at lower intensity when the driving field is on-resonance, compared to when it is off resonance. As discussed in the section on Rate Equation equations, the net result is that the spectral line shape remains Lorentzian but becomes much broader, which makes it harder to find the line center. We can manage or eliminate power broadening by keeping the laser intensity well below the saturation intensity for the given transition.

Problem IV

Problem IV

(a) We have (following the lecture notes on the Quantum Beam Splitter)

$$|\Psi_{in}\rangle = |1\rangle_1|1\rangle_2 = \hat{a}_1^+\hat{a}_2^+|0\rangle \Rightarrow$$

$$|\Psi_{out}\rangle = \frac{1}{\sqrt{2}}\left(\hat{a}_3^+ + i\hat{a}_4^+\right)\left(i\hat{a}_3^+ + \hat{a}_4^+\right)|0\rangle = \frac{i}{2}\left(\hat{a}_3^+\hat{a}_3^+ + \hat{a}_4^+\hat{a}_4^+\right)|0\rangle = \frac{i}{\sqrt{2}}\left(|2\rangle_3|0\rangle_4 + |0\rangle_3|2\rangle_4\right)$$

Because $|\Psi_{out}\rangle$ has zero probability amplitude for the $|1\rangle_3|1\rangle_4$ state the probability of coincidence detection is zero.

(b) Here we have

$$|\Psi_{in}\rangle = \frac{1}{\sqrt{2}}\hat{a}_1^+\hat{a}_1^+|0\rangle = |2\rangle_1|0\rangle_2 = \frac{1}{\sqrt{2}}\left(t\hat{a}_3^+ + r\hat{a}_4^+\right)\left(t\hat{a}_3^+ + r\hat{a}_4^+\right)|0\rangle$$

$$= \frac{1}{\sqrt{2}}\left(t^2\hat{a}_3^+\hat{a}_3^+ - r^2\hat{a}_4^+\hat{a}_4^+ + i2tr\hat{a}_3^+\hat{a}_4^+\right)|0\rangle = \frac{1}{2}\left(\frac{1}{\sqrt{2}}\hat{a}_3^+\hat{a}_3^+ - \frac{1}{\sqrt{2}}\hat{a}_4^+\hat{a}_4^+ + 2i\frac{1}{\sqrt{2}}\hat{a}_3^+\hat{a}_4^+\right)|0\rangle$$

$$= \left(\frac{1}{2}|2\rangle_3|0\rangle_4 - \frac{1}{2}|0\rangle_3|2\rangle_4 + \frac{i}{\sqrt{2}}|1\rangle_3|1\rangle_4\right) \Rightarrow$$

$$|\Psi_{out}\rangle = \left(\frac{1}{2}|2\rangle_3|0\rangle_4 - \frac{1}{2}|0\rangle_3|2\rangle_4 + \frac{i}{\sqrt{2}}|1\rangle_3|1\rangle_4\right)$$

From the probability amplitude for the state $|1\rangle_3|1\rangle_4$ we see that the probability of a coincidence detection is 1/2.

(c) We have

$$|\Psi_{in}\rangle = \frac{1}{\sqrt{2}}\left(|2\rangle_1|0\rangle_2 + |0\rangle_1|2\rangle_2\right) = \frac{1}{2}\left(\hat{a}_1^+\hat{a}_1^+ + \hat{a}_2^+\hat{a}_2^+\right)|0\rangle \Rightarrow$$

$$|\Psi_{out}\rangle = \frac{1}{4}\left[\left(\hat{a}_3^+ + i\hat{a}_4^+\right)\left(\hat{a}_3^+ + i\hat{a}_4^+\right) + \left(i\hat{a}_3^+ + \hat{a}_4^+\right)\left(i\hat{a}_3^+ + \hat{a}_4^+\right)\right]|0\rangle$$

$$= \frac{1}{4}\left(\hat{a}_3^+\hat{a}_3^+ - \hat{a}_4^+\hat{a}_4^+ + i2\hat{a}_3^+\hat{a}_4^+ - \hat{a}_3^+\hat{a}_3^+ + \hat{a}_4^+\hat{a}_4^+ + i2\hat{a}_3^+\hat{a}_4^+\right)|0\rangle$$

$$= i\hat{a}_3^+\hat{a}_4^+|0\rangle = i|1\rangle_3|1\rangle_4$$

Because $|\Psi_{out}\rangle$ has a probability amplitude with unit norm for the $|1\rangle_3|1\rangle_4$ state the probability of coincidence is one.