OPTI 544 Solution Set 7, Spring 2024

Problem 1

(a) From the Notes/Slides "Quantum Electrodynamics" we have the classical Hamiltonian and Lagrangian,

$$\mathcal{H} = \sum_j \frac{1}{2} m_j \dot{q}_j^2 + \sum_j \frac{1}{2} m_j \omega_j^2 q_j^2$$

$$\mathcal{L} = \sum_j \frac{1}{2} m_j \dot{q}_j^2 - \sum_j \frac{1}{2} m_j \omega_j^2 q_j^2$$

The Lagrange Equation of motion is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = m_j \ddot{q}_j + m_j \omega_j^2 q_j = 0 \quad \Rightarrow \quad \ddot{q}_j + \omega_j^2 q_j = 0$$

(b) The normal mode expansion of the field is $E_x(z,t) = \sum_j A_j q_j(t) \sin(k_j z)$

Substitute in the wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \sum_j A_j q_j(t) \sin(k_j z) = \sum_j -A_j k_j^2 q_j(t) \sin(k_j z) - \frac{1}{c^2} \ddot{q}_j \sin(k_j z) = 0$$

This must hold independently for each j. Using $k_j = \frac{\omega_j}{c}$ we have

$$\ddot{q}_j + \omega_j^2 q_j = 0$$

This is identical to the result from using the Lagrange formalism in (a) above. Thus, the Lagrangian and the wave equation have the same dynamical eigenmodes governed by the same equations of motion, the two must be equivalent. That in turn tells us that we have found the correct Lagrangian.

Problem 2

(a) We can write the input as

$$\left|\Psi_{in}\right\rangle = \left(\sqrt{1-\varepsilon}\,\hat{a}_1^{\dagger} + \sqrt{\varepsilon/2}\,\hat{a}_1^{\dagger 2}\right)\left(\sqrt{1-\varepsilon}\,\hat{a}_2^{\dagger} + \sqrt{\varepsilon/2}\,\hat{a}_2^{\dagger 2}\right)\left|0\right\rangle$$

Substituting $\hat{a}_1^{\dagger} = \frac{1}{\sqrt{2}}\left(\hat{a}_3^{\dagger} + i\hat{a}_4^{\dagger}\right)$ and $\hat{a}_2^{\dagger} = \frac{1}{\sqrt{2}}\left(i\hat{a}_3^{\dagger} + \hat{a}_4^{\dagger}\right)$ we get

$$\left|\Psi_{out}\right\rangle = \left[\frac{i}{2}(1-\varepsilon)\left(i\hat{a}_3^{\dagger 2} + \hat{a}_4^{\dagger 2}\right) - \frac{1-i}{4}\sqrt{\varepsilon(1-\varepsilon)}\left(\hat{a}_3^{\dagger 3} + \hat{a}_4^{\dagger 3} + \hat{a}_3^{\dagger}\hat{a}_4^{\dagger 2} + \hat{a}_3^{\dagger 2}\hat{a}_4^{\dagger}\right) - \frac{\varepsilon}{8}\left(\hat{a}_3^{\dagger 4} + \hat{a}_4^{\dagger 4}\right) - \frac{\varepsilon}{4}\left(\hat{a}_3^{\dagger 2}\hat{a}_4^{\dagger 3}\right)\right]\left|0\right\rangle$$

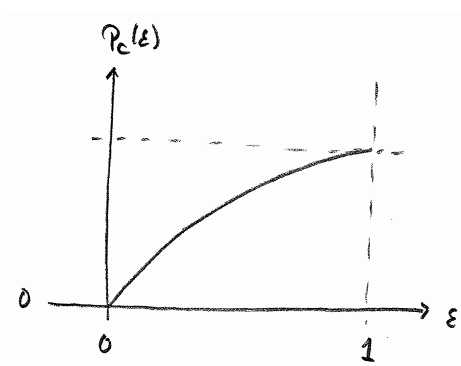
$$= \frac{i}{\sqrt{2}}(1-\varepsilon)\left(|2\rangle_3|0\rangle_4 + |0\rangle_3|2\rangle_4\right) - \frac{1-i}{4}\sqrt{6\varepsilon(1-\varepsilon)}\left(|3\rangle_3|0\rangle_4 + |0\rangle_3|3\rangle_4\right)$$
$$- \frac{1-i}{4}\sqrt{2\varepsilon(1-\varepsilon)}\left(|1\rangle_3|2\rangle_4 + |2\rangle_3|1\rangle_4\right) - \frac{\sqrt{6}\,\varepsilon}{4}\left(|4\rangle_3|0\rangle_4 + |0\rangle_3 4_4\right) - \frac{\varepsilon}{2}|2\rangle_3|2\rangle_4$$

The output is a superposition of number states with various combinations of photon numbers in each port. The probability of a coincidence detection is the sum of the probability amplitudes squared for all the states that have at least one photon in each port. This gives us

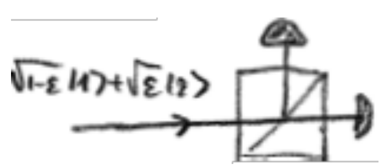
$$P_c = \left(\frac{\varepsilon}{2}\right)^2 + 2\times\left|\frac{1-i}{4}\sqrt{2\varepsilon(1-\varepsilon)}\right|^2 = \frac{1}{4}\varepsilon(2-\varepsilon)$$

(b) P_c takes on a maximum value of $1/4$ at $\varepsilon = 1$. For $\varepsilon \ll 1$ we have $P_c \approx \frac{1}{2}\varepsilon$.

Sketch:



Note: To check for two-photon admixture we could just input a single pulse through one port,



Here $P_c \approx \frac{1}{2}\varepsilon$