OPTI 544 Solution Set 8, Spring 2024

Problem 1

(a) The input state to the Beamsplitter is $|\Psi_{in}\rangle = e^{\alpha\hat{a}_1^+ - \alpha^*\hat{a}_1}|0\rangle$, where $|0\rangle$ is the two-mode vacuum. We use $\hat{a}_1^+ = t\hat{a}_3^+ + r\hat{a}_4^+$ and its H. C. to find the state after the 1st BS:

$$\left|\Psi'_{out}\right\rangle = e^{(t\alpha)\hat{a}_3^+ - (t\alpha)^*\hat{a}_3 + (r\alpha)\hat{a}_4^+ - (r\alpha)^*\hat{a}_4}|0\rangle$$

Before the 2nd BS we have

$$\left|\Psi'_{in}\right\rangle = e^{(t\alpha e^{i\varphi_1})\hat{a}_3^+ - (t\alpha e^{i\varphi_1})^*\hat{a}_3 + (r\alpha e^{i\varphi_2})\hat{a}_4^+ - (r\alpha e^{i\varphi_2})^*\hat{a}_4}|0\rangle$$

Next, for the 2nd BS we use $\hat{a}_3^+ = t\hat{a}_5^+ + r\hat{a}_6^+$, $\hat{a}_4^+ = r\hat{a}_5^+ + t\hat{a}_6^+$ to get the output state,

$$\begin{aligned}
|\Psi_{out}\rangle &= \exp[(t\alpha e^{i\varphi_1})(t\hat{a}_5^+ + r\hat{a}_6^+) - (t\alpha e^{i\varphi_1})^*(t^*\hat{a}_5 + r^*\hat{a}_6) \\
&\qquad + (r\alpha e^{i\varphi_2})(r\hat{a}_5^+ + t\hat{a}_6^+) - (r\alpha e^{i\varphi_2})^*(r^*\hat{a}_5 + t^*\hat{a}_6)]|0\rangle \\
&= \exp[(t^2\alpha e^{i\varphi_1} + r^2\alpha e^{i\varphi_2})\hat{a}_5^+ - (t^2\alpha e^{i\varphi_1} + r^2\alpha e^{i\varphi_2})^*\hat{a}_5 \\
&\qquad + (rt\alpha e^{i\varphi_1} + rt\alpha e^{i\varphi_2})\hat{a}_6^+ - (rt\alpha e^{i\varphi_1} + rt\alpha e^{i\varphi_2})^*\hat{a}_6]|0\rangle \\
&= |\alpha_5\rangle|\alpha_6\rangle
\end{aligned}$$

Setting $t = 1/\sqrt{2}$, $r = i/\sqrt{2}$ we find

$$\begin{aligned}
\alpha_5 &= \alpha(t^2 e^{i\varphi_1} + r^2 e^{i\varphi_2}) = \frac{\alpha}{2}(e^{i\varphi_1} - e^{i\varphi_2}) = \frac{\alpha}{2}e^{i(\varphi_1+\varphi_2)/2}(e^{i(\varphi_1-\varphi_2)/2} - e^{-i(\varphi_1-\varphi_2)/2}) \\
&= i\alpha e^{i\varphi_0}\sin(\delta\varphi/2 - \pi/4) = \frac{i\alpha}{\sqrt{2}}e^{i\varphi_0}[\sin(\delta\varphi/2) - \cos(\delta\varphi/2)]
\end{aligned}$$

$$\begin{aligned}
\alpha_6 &= i\frac{\alpha}{2}(e^{i\varphi_1} - e^{i\varphi_2}) = i\frac{\alpha}{2}e^{i(\varphi_1+\varphi_2)/2}(e^{i(\varphi_1-\varphi_2)/2} + e^{-i(\varphi_1-\varphi_2)/2}) \\
&= i\alpha e^{i\varphi_0}\cos(\delta\varphi/2 - \pi/4) = \frac{i\alpha}{\sqrt{2}}e^{i\varphi_0}[\sin(\delta\varphi/2) + \cos(\delta\varphi/2)]
\end{aligned}$$

(b) We have

$$\begin{aligned}
\langle\hat{S}\rangle &= \langle\alpha_5|\langle\alpha_6|\,\hat{a}_6^+\hat{a}_6 - \hat{a}_5^+\hat{a}_5\,|\alpha_6\rangle|\alpha_5\rangle \\
&= \frac{|\alpha|^2}{2}[\{\sin(\delta\varphi/2) + \cos(\delta\varphi/2)\}^2 - \{\sin(\delta\varphi/2) - \cos(\delta\varphi/2)\}^2] \\
&= \frac{|\alpha|^2}{2}[4\sin(\delta\varphi/2)\cos(\delta\varphi/2)] = |\alpha|^2\sin(\delta\varphi) \approx |\alpha|^2\delta\varphi
\end{aligned}$$

(c) First we compute

$$\langle \hat{S}^2 \rangle = \langle \alpha_5 | \langle \alpha_6 | (\hat{a}_6^+ \hat{a}_6 - \hat{a}_5^+ \hat{a}_5)^2 | \alpha_6 \rangle | \alpha_5 \rangle$$

$$= \langle \hat{a}_6^+ \hat{a}_6 \hat{a}_6^+ \hat{a}_6 + \hat{a}_5^+ \hat{a}_5 \hat{a}_5^+ \hat{a}_5 - 2\hat{a}_6^+ \hat{a}_6 \hat{a}_5^+ \hat{a}_5 \rangle \qquad \text{(use } \hat{a}\hat{a}^+ = \hat{a}^+\hat{a} + 1\text{)}$$

$$= \langle \hat{a}_6^+ \hat{a}_6^+ \hat{a}_6 \hat{a}_6 + \hat{a}_5^+ \hat{a}_5^+ \hat{a}_5 \hat{a}_5 + \hat{a}_6^+ \hat{a}_6 + \hat{a}_5^+ \hat{a}_5 - 2\hat{a}_6^+ \hat{a}_6 \hat{a}_5^+ \hat{a}_5 \rangle$$

$$= |\alpha_6|^4 + |\alpha_5|^4 + |\alpha_6|^2 + |\alpha_5|^2 - 2|\alpha_6|^2 |\alpha_5|^2$$

Then, using $\langle \hat{S} \rangle^2 = (|\alpha_6|^2 - |\alpha_5|^2)^2 = |\alpha_6|^4 + |\alpha_5|^4 - 2|\alpha_6|^2 |\alpha_5|^2$, we find

$$\Delta S^2 = \langle \hat{S}^2 \rangle - \langle \hat{S} \rangle^2 = |\alpha_6|^2 + |\alpha_5|^2$$

$$= |\alpha|^2 [\cos^2(\delta\varphi/2 - \pi/4) + \sin^2(\delta\varphi/2 - \pi/4)] = |\alpha|^2$$

(d) We set $\boxed{|\alpha|^2 \delta\varphi_{\min} = |\alpha| \Rightarrow \delta\varphi_{\min} = \frac{1}{|\alpha|} = \frac{1}{\sqrt{\overline{n}}}}$ where $\overline{n} = |\alpha|^2$ is the mean photon number in a coherent state with amplitude $|\alpha|$.

This is the *shot-noise limited* sensitivity of a Mach-Zender interferometer with a coherent state input. In quantum metrology, this is also referred to as the *standard quantum limit.* Improved sensitivity can be achieved with squeezed states or other non-classical states of light.

Note: This was a very long and fiddly calculation with many opportunities to mess up the math, especially if not knowing ahead of time how it was supposed to come out. However, the final result is a very important example of applied quantum optics.

Problem 2

(a) We have $\hat{H}(t) = \hbar\omega\,(\hat{a}^+\hat{a} + 1/2) + \hbar\lambda(t)\,(\hat{a} + \hat{a}^+)$.

We need the commutators

$$[\hat{a},\hat{a}^+] = 1 \qquad [\hat{a},\hat{a}^+\hat{a}] = \hat{a}$$

$$[\hat{a}^+,\hat{a}] = -1 \qquad [\hat{a}^+,\hat{a}^+\hat{a}] = -\,\hat{a}^+$$

From these it follows that

$$[\hat{a},\hat{H}(t)] = \hbar\omega\hat{a} + \hbar\lambda(t)$$

$$[\hat{a}^+,\hat{H}(t)] = -\hbar\omega\hat{a}^+ - \hbar\lambda(t)$$

(b) Let $\alpha(t) = \langle\psi(t)|\,\hat{a}\,|\psi(t)\rangle$. Then (Cohen-Tannoudji p. 240)

$$\frac{d}{dt}\alpha(t) = \frac{1}{i\hbar}\langle[\hat{a},\hat{H}(t)]\rangle = \frac{1}{i\hbar}(\hbar\omega\langle\hat{a}\rangle + \hbar\lambda(t)) = -i\omega\alpha(t) - i\lambda(t)$$

This equation can be integrated to give

$$\alpha(t) = e^{-i\omega t}\int_{t_0}^{t} e^{i\omega t'}[-i\lambda(t')]\,dt' + \alpha(t_0)e^{-i\omega(t-t_0)} \qquad \text{for} \qquad \lambda \neq 0$$

$$\alpha(t) = \alpha(t_0)e^{-i\omega(t-t_0)} \qquad \text{for} \qquad \lambda = 0$$

For the quadrature operators we have

$$\langle\hat{X}\rangle(t) = \frac{1}{2}(\alpha + \alpha^*) = \mathrm{Re}[\alpha]$$

$$\langle\hat{Y}\rangle(t) = \frac{1}{2i}(\alpha - \alpha^*) = \mathrm{Im}[\alpha]$$

Because we cannot do the integral in the expression for $\alpha(t)$ until the specific form of $\lambda(t)$ is known, this is the best we can do.

(c) Let $|\varphi(t)\rangle = [\hat{a} - \alpha(t)]|\psi(t)\rangle$. Then

$$i\hbar\frac{d}{dt}|\varphi(t)\rangle = i\hbar\frac{d}{dt}\hat{a}|\psi(t)\rangle - i\hbar\frac{d}{dt}[\alpha(t)|\psi(t)\rangle]$$

$$= i\hbar\hat{a}\frac{d}{dt}|\psi(t)\rangle - i\hbar[-i\omega\alpha(t)|\psi(t)\rangle - i\lambda|\psi(t)\rangle] - i\hbar\alpha(t)\frac{d}{dt}|\psi(t)\rangle$$

$$= i\hbar[\hat{a} - \alpha(t)]\frac{d}{dt}|\psi(t)\rangle + [-\hbar\omega\alpha(t) - \hbar\lambda(t)]|\psi(t)\rangle$$

Now

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle \;\Rightarrow$$
$$i\hbar\frac{d}{dt}|\psi(t)\rangle = i\hbar[\hat{a}-\alpha(t)]\frac{1}{i\hbar}\hat{H}(t)|\psi(t)\rangle - \hbar[\omega\alpha(t)+\lambda(t)]|\psi(t)\rangle$$

The commutator $[\hat{a},\hat{H}(t)] = \hbar\omega\hat{a}+\hbar\lambda(t)$, so $\hat{a}\hat{H}(t) = \hat{H}(t)\hat{a}+\hbar\omega\hat{a}+\hbar\lambda(t)$.

Putting this together gives us

$$i\hbar\frac{d}{dt}|\varphi(t)\rangle = [\hat{H}(t)\hat{a}+\hbar\omega\hat{a}-\hat{H}\alpha(t)+\hbar\lambda(t)]|\psi(t)\rangle - [\hbar\omega\alpha(t)+\hbar\lambda(t)]|\psi(t)\rangle$$
$$= \hat{H}[\hat{a}-\alpha(t)]|\psi(t)\rangle + \hbar\omega[\hat{a}-\alpha(t)]|\psi(t)\rangle = (\hat{H}+\hbar\omega)|\psi(t)\rangle$$

Now $$\frac{d}{dt}\langle\varphi(t)|\varphi(t)\rangle = [\frac{d}{dt}\langle\varphi(t)|]|\varphi(t)\rangle + \langle\varphi(t)|\frac{d}{dt}|\varphi(t)\rangle$$
$$= \langle\varphi(t)|\frac{\hat{H}+\hbar\omega}{i\hbar}|\varphi(t)\rangle + \langle\varphi(t)|\frac{\hat{H}+\hbar\omega}{-i\hbar}|\varphi(t)\rangle \equiv 0$$

Conclusion: $\|\varphi(t)\rangle\|$ is preserved over time.

(d) We have $\hat{a}|\psi(0)\rangle = \alpha(0)|\psi(t)\rangle \;\Rightarrow |\varphi(0)\rangle \equiv 0$

Since $\|\varphi(t)\rangle\|$ is preserved, it follows that $\|\varphi(t)\rangle\| = 0$ and therefore

$$\hat{a}|\psi(t)\rangle = \alpha(t)|\psi(t)\rangle$$

Note we already have an expression for $\alpha(t)$ in terms of an integral that involves $\lambda(t)$.

(e) At $t=0$ we have $|\psi(t)\rangle = |\varphi(0)\rangle = |\alpha(0)\rangle$, with $\alpha(0)\equiv 0$.

We start in the coherent state $|\alpha(0)\rangle$, then have $\lambda(t)\neq 0$ for $t\in[0,T]$. During this interval we still have a coherent state, but $\alpha(t)$ is changing. At time T we have

$$\alpha(T) = -ie^{-i\omega T}\int_0^T e^{i\omega t'}\lambda(t')dt'$$

At $t > T$ we also have a coherent state, with

$$\alpha(t) = \alpha(T)e^{-i\omega(t-T)}$$