The Interaction Picture

Before we discuss the differential equation for $c_n(t)$, we discuss the interaction picture. Suppose we have a physical system such that its state ket coincides with $|\alpha\rangle$ at $t=t_0$, where t_0 is often taken to be zero. At a later time, we denote the state ket in the Schrödinger picture by $|\alpha, t_0; t\rangle_S$, where the subscript S reminds us that we are dealing with the state ket of the Schrödinger picture.

We now *define*

$$|\alpha, t_0; t\rangle_I = e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S, \tag{5.5.5}$$

where $|\,\rangle_I$ stands for a state ket that represents the same physical situation in the *interaction picture*. At $t=0$, $|\,\rangle_I$ evidently coincides with $|\,\rangle_S$. For operators (representing observables) we define observables in the interaction picture as

$$A_I \equiv e^{iH_0t/\hbar}A_S e^{-iH_0t/\hbar}. \tag{5.5.6}$$

In particular,

$$V_I = e^{iH_0t/\hbar}Ve^{-iH_0t/\hbar} \tag{5.5.7}$$

where V without a subscript is understood to be the time-dependent potential in the Schrödinger picture. The reader may recall here the connection between the Schrödinger picture and the Heisenberg picture:

$$|\alpha\rangle_H = e^{+iHt/\hbar}|\alpha, t_0=0; t\rangle_S \tag{5.5.8}$$

$$A_H = e^{iHt/\hbar}A_S e^{-iHt/\hbar}. \tag{5.5.9}$$

The basic difference between (5.5.8) and (5.5.9) on the one hand and (5.5.6) and (5.5.7) on the other is that H rather than H_0 appears in the exponential.

We now derive the fundamental differential equation that characterizes the time evolution of a state ket in the interaction picture. Let us take the time derivative of (5.5.5) with the full H given by (5.5.1):

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle_I &= i\hbar\frac{\partial}{\partial t}\left(e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S\right) \\ &= -H_0e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S + e^{iH_0t/\hbar}(H_0+V)|\alpha, t_0; t\rangle_S \\ &= e^{iH_0t/\hbar}Ve^{-iH_0t/\hbar}e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S. \end{aligned} \tag{5.5.10}$$

We thus see

$$i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle_I = V_I|\alpha, t_0; t\rangle_I, \tag{5.5.11}$$

which is a Schrödinger-like equation with the total H replaced by V_I. In other words $|\alpha, t_0; t\rangle_I$ would be a ket fixed in time if V_I were absent. We can also show for an observable A (that does not contain time t explicitly in the Schrödinger picture) that

$$\frac{dA_I}{dt} = \frac{1}{i\hbar}[A_I, H_0], \tag{5.5.12}$$

which is a Heisenberg-like equation with H replaced by H_0.

TABLE 5.2

	Heisenberg picture	Interaction picture	Schrödinger picture
State ket	No change	Evolution determined by V_I	Evolution determined by H
Observable	Evolution determined by H	Evolution determined by H_0	No change

In many respects, the interaction picture, or *Dirac picture*, is intermediate between the Schrödinger picture and the Heisenberg picture; This should be evident from Table 5.2.

In the interaction picture we continue using $|n\rangle$ as our base kets. Thus we expand $|\,\rangle_I$ as follows:

$$|\alpha, t_0; t\rangle_I = \sum_n c_n(t)|n\rangle. \tag{5.5.13}$$

With t_0 set equal to 0, we see that the $c_n(t)$ appearing here are the same as the $c_n(t)$ introduced earlier in (5.5.4), as can easily be verified by multiplying both sides of (5.5.4) by $e^{iH_0t/\hbar}$ using (5.5.2).

We are finally in a position to write the differential equation for $c_n(t)$. Multiplying both sides of (5.5.11) by $\langle n|$ from the left, we obtain

$$i\hbar\frac{\partial}{\partial t}\langle n|\alpha, t_0; t\rangle_I = \sum_m \langle n|V_I|m\rangle\langle m|\alpha, t_0; t\rangle_I. \tag{5.5.14}$$

This can also be written using

$$\langle n|e^{iH_0t/\hbar}V(t)e^{-iH_0t/\hbar}|m\rangle = V_{nm}(t)e^{i(E_n-E_m)t/\hbar}$$

and

$$c_n(t) = \langle n|\alpha, t_0; t\rangle_I$$

[from (5.5.13)] as

$$i\hbar\frac{d}{dt}c_n(t) = \sum_m V_{nm}e^{i\omega_{nm}t}c_m(t), \tag{5.5.15}$$

where

$$\omega_{nm} \equiv \frac{(E_n - E_m)}{\hbar} = -\omega_{mn}. \tag{5.5.16}$$

Explicitly,

$$i\hbar\begin{pmatrix}\dot{c}_1\\ \dot{c}_2\\ \dot{c}_3\\ \vdots\end{pmatrix} = \begin{pmatrix} V_{11} & V_{12}e^{i\omega_{12}t} & & \cdots \\ V_{21}e^{i\omega_{21}t} & V_{22} & & \cdots \\ & & V_{33} & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\\ \vdots\end{pmatrix}. \tag{5.5.17}$$

This is the basic coupled differential equation that must be solved to obtain the probability of finding $|n\rangle$ as a function of t.