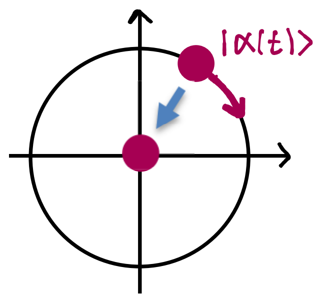
Quantum States of the Quantized Field

Odds and Ends – Quantum-Classical Correspondence

Define a Translation Operator

$$\hat T_\alpha(t) = e^{\alpha^* e^{i\omega t}\hat a - \alpha e^{-i\omega t}\hat a^\dagger} = \hat D(-\alpha e^{-i\omega t})$$

This translates a coherent state to the vacuum



We can use the relation

$$[\hat a, \hat F(\hat a^\dagger)] = dF(\hat a^\dagger)/d\hat a^\dagger$$

to show that

$$[\hat a, \hat T_\alpha] = \hat a \hat T_\alpha - \hat T_\alpha \hat a = -\alpha e^{-i\omega t} \hat T_\alpha$$

We can rearrange and show that

$$\hat T_\alpha \hat a = \hat a \hat T_\alpha + \alpha e^{-i\omega t} \hat T_\alpha$$

$$\Rightarrow \hat T_\alpha \hat a \hat T_\alpha^\dagger = \underbrace{\hat a}_{\text{Field Observable}} + \underbrace{\alpha e^{-i\omega t}}_{\text{C-valued dimensionless field amplitude}}$$

Field Observable

C-valued dimensionless field amplitude

From this we get

$$\underbrace{\hat T_\alpha \hat E_\perp \hat T_\alpha^\dagger}_{\hat E_\perp'} = \hat T_\alpha\left(\mathcal{E}_k \hat a e^{i\vec k\cdot\vec r} + H.C.\right)\hat T_\alpha^\dagger$$

$$+ \mathcal{E}_k \alpha e^{-i(\omega t - \vec k\cdot\vec r)} + C.C.$$

New Field Observable

$E_\perp^{Cl}(\alpha, t)$ Classical Field

Overall, the unitary transformation $\hat T_\alpha(t)$ implements the map

$$\boxed{\begin{gathered}\hat T_\alpha(t)\hat E_\perp \hat T_\alpha(t)^\dagger = \hat E_\perp + E_\perp^{Cl}(\alpha, t) \\ |\psi'(t)\rangle = T_\alpha(t)|\alpha(t)\rangle = |0\rangle\end{gathered}}$$

We can work with

$\hat E_\perp, |\alpha(t)\rangle$ **or** $\hat E_\perp + E_\perp^{Cl}(\alpha, t)$, $|0\rangle$

Validates Semiclassical Optics for strong Coherent Fields!

Quantized Light – Matter Interactions

Quantized Light – Matter Interactions

General Problem:

Quantum Atom	⟷	Quantized Modes Of EM Field

Electric Dipole Interaction

Starting Point: System Hamiltonian

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} \qquad (1)$$

$$\hat{H}_F = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \left(\hat{a}^+_{\vec{k}}\hat{a}_{\vec{k}} + 1/2\right) \qquad \text{Field}$$

$$\hat{H}_A = \sum_i E_i |i\rangle\langle i| = \sum_i E_i \hat{\sigma}_{ii} \qquad \text{Atom}$$

$$\hat{H}_{AF} = -\hat{\vec{\mu}} \cdot \hat{\vec{E}}(\vec{r},t) \qquad \text{ED interaction}$$

E_i, $|i\rangle$: energies, energy levels of the atom

Dipole Operator:

$$(2) \quad \boxed{\hat{\vec{\mu}} = \sum_{i,j} \vec{\mu}_{ij} |i\rangle\langle j| = \sum_{i,j} \vec{\mu}_{ij}\hat{\sigma}_{ij}}$$

Field Operator:

$$\vec{E}(\vec{r},t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + H.C., \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar\omega_{\vec{k}}}{2\varepsilon_0 V}}$$

2 polarization modes implicit

Pin down atom where $u_{\vec{k}}(\vec{r}) = 1$

– anywhere if $u_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$

– if $u_{\vec{k}}(\vec{r}) = \sin(kz)$ then where $\sin(kz) = 1$

$$(3) \quad \boxed{\hat{\vec{E}}(\vec{r},t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \left(\hat{a}_{\vec{k}} + \hat{a}^+_{\vec{k}}\right)}$$

Quantized Light – Matter Interactions

Dipole Operator:

$$\hat{\vec{\mu}} = \sum_{i,j} \vec{\mu}_{ij} |i\rangle\langle j| = \sum_{i,j} \vec{\mu}_{ij} \hat{\sigma}_{ij} \tag{2}$$

Field Operator:

$$\vec{E}(\vec{r},t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + \text{H.C.}, \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar\omega_{\vec{k}}}{2\varepsilon_0 V}}$$

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Pin down atom where $u_{\vec{k}}(\vec{r}) = 1$

- anywhere if $u_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$
- if $u_{\vec{k}}(\vec{r}) = \sin(kz)$ then where $\sin(kz) = 1$

$$\hat{\vec{E}}(\vec{r},t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}}) \tag{3}$$

Combining (2) & (3):

$$\hat{H}_{AF} = \sum_{ij} \sum_{\vec{k}} -\vec{\mu}_{ij} \cdot \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})$$
$$= \sum_{ij} \sum_{\vec{k}} \hbar g^{(ij)}_{\vec{k}} \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})$$

where $\hbar g^{(ij)}_{\vec{k}} = \vec{\mu}_{ij} \cdot \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}}$

Rabi Freq., same sign convention as when we first looked at the Rabi problem

2-level atom ⇨ $(i,j) = (1,2)$:

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} (g_{\vec{k}} \hat{\sigma}_{21} + g^{*}_{\vec{k}} \hat{\sigma}_{12}) (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})$$

Define:

$$\left.\begin{aligned} \hat{\sigma}_{+} &= \hat{\sigma}_{21} = |2\rangle\langle 1| \\ \hat{\sigma}_{-} &= \hat{\sigma}_{12} = |1\rangle\langle 2| \\ \hat{\sigma}_{z} &= \hat{\sigma}_{22} - \hat{\sigma}_{11} = |2\rangle\langle 2| - |1\rangle\langle 1| \end{aligned}\right\} \Rightarrow$$

Pauli matrices

$$\hat{\sigma}_{x} = \tfrac{1}{2}(\hat{\sigma}_{+} + \hat{\sigma}_{-})$$
$$\hat{\sigma}_{y} = \tfrac{1}{2i}(\hat{\sigma}_{+} - \hat{\sigma}_{-})$$
$$\hat{\sigma}_{z}$$

Quantized Light – Matter Interactions

Combining (2) & (3):

$$\hat{H}_{AF} = \sum_{ij}\sum_{\vec{k}} -\vec{\eta}_{ij}\cdot\vec{\varepsilon}_{\vec{k}}\,\mathcal{E}_{\vec{k}}\,\hat{\sigma}_{ij}\,(\hat{a}_{\vec{k}}+\hat{a}^{+}_{\vec{k}})$$

$$= \sum_{ij}\sum_{\vec{k}} \hbar g^{(ij)}_{\vec{k}}\,\hat{\sigma}_{ij}\,(\hat{a}_{\vec{k}}+\hat{a}^{+}_{\vec{k}})$$

where $\hbar g^{(ij)}_{\vec{k}} = \vec{\eta}_{ij}\cdot\vec{\varepsilon}_{\vec{k}}\,\mathcal{E}_{\vec{k}}$

Rabi Freq., same sign convention as when we first looked at the Rabi problem

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$$\hat{H}_{AF} = \hbar\sum_{\vec{k}}\left(g_{\vec{k}}\hat{\sigma}_{21} + g^{*}_{\vec{k}}\hat{\sigma}_{12}\right)(\hat{a}_{\vec{k}}+\hat{a}^{+}_{\vec{k}})$$

Define:

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$$\hat{\sigma}_{z} = \hat{\sigma}_{22} - \hat{\sigma}_{11} = |2\rangle\langle 2| - |1\rangle\langle 1|$$

Pauli matrices

$$\hat{\sigma}_{x} = \frac{1}{2}(\hat{\sigma}_{+}+\hat{\sigma}_{-})$$

$$\hat{\sigma}_{y} = \frac{1}{2i}(\hat{\sigma}_{+}-\hat{\sigma}_{-})$$

$$\hat{\sigma}_{z}$$

With this notation **(4)**

$$\hat{H}_{AF} = \hbar\sum_{\vec{k}}\left(g_{\vec{k}}\hat{\sigma}_{+}\hat{a}_{\vec{k}} + \cancel{g_{\vec{k}}\hat{\sigma}_{+}\hat{a}^{+}_{\vec{k}}} + \cancel{g_{\vec{k}}\hat{\sigma}_{-}\hat{a}_{\vec{k}}} + g_{\vec{k}}\hat{\sigma}_{-}\hat{a}^{+}_{\vec{k}}\right)$$

Energy conservation?

$$\hat{H}_{AF} = \hbar\sum_{\vec{k}}\left(g_{\vec{k}}\hat{\sigma}_{+}\hat{a}_{\vec{k}} + g_{\vec{k}}\hat{\sigma}_{-}\hat{a}^{+}_{\vec{k}}\right)$$

Putting it all together

(5)

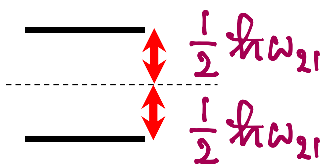
$$\hat{H} = \hat{H}_{F} + \hat{H}_{A} + \hat{H}_{AF} =$$

$$\sum_{\vec{k}}\hbar\omega_{\vec{k}}\hat{a}^{+}_{\vec{k}}\hat{a}_{\vec{k}} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_{z} + \sum_{\vec{k}}\hbar\left(g_{\vec{k}}\hat{\sigma}_{+}\hat{a}_{\vec{k}} + g_{\vec{k}}\hat{\sigma}_{-}\hat{a}^{+}_{\vec{k}}\right)$$

We changed the zero point for energy by subtracting

$\sum_{\vec{k}}\frac{1}{2}\hbar\omega_{\vec{k}}$ and $\frac{1}{2}(E_2 - E_1)$

field atom



Quantized Light – Matter Interactions

With this notation (4)

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + \cancel{g_{\vec{k}} \hat{\sigma}_+ \hat{a}^+_{\vec{k}}} + \cancel{g_{\vec{k}} \hat{\sigma}_- \hat{a}_{\vec{k}}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right)$$

Energy conservation?

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right)$$

Putting it all together

$$\boxed{\begin{aligned} &\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} = \qquad (5) \\ &\sum_{\vec{k}} \hbar\omega_{\vec{k}} \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_z + \sum_{\vec{k}} \hbar \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right) \end{aligned}}$$

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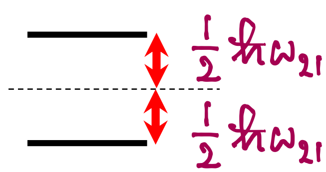


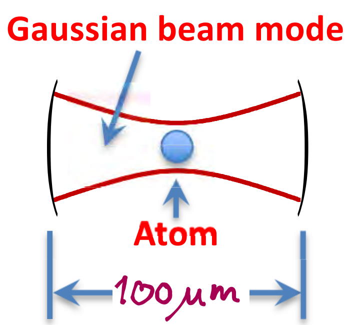
field **atom**

Foundational result for remainder of course

Quantized Light – Matter Interactions

Interaction with Single-mode Fields

Good approx. in small, high-Q Cavity



$c/2L \gg A_{21}$

$|g_{\vec{k}}| \gg A_{21}, \gamma$

Single-mode (Jaynes-Cummings) Hamiltonian

$$\boxed{\hat{H} = \underbrace{\hbar\omega\,\hat{a}^{+}\hat{a} + \tfrac{1}{2}\hbar\omega_{21}\hat{\sigma}_{z}}_{H_0} + \underbrace{\hbar g(\hat{\sigma}_{+} + \hat{\sigma}_{-})(\hat{a} + \hat{a}^{+})}_{H_{AF}}}$$

Paradigm for spin-1/2 coupled to QHO

- Atom in high-Q cavity *)
- Quantum dot in high-Q
- Rydberg atom in superconducting μw Cavity
- Superconducting qubit in superconducting μw cavity
- Superconducting qubit in superconducting μw stripline cavity (circuit QED)
- Trapped ion with quantized COM motion *)

*) Nobel Prize in Physics 2012

Quantized Light – Matter Interactions

Paradigm for spin-1/2 coupled to QHO

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More about Single-mode Cavity QED

$$\hat{H} = \underbrace{\hbar\omega\,\hat{a}^{+}\hat{a} + \tfrac{1}{2}\hbar\omega_{21}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar\left(g_{\vec{k}}\hat{\sigma}_{+} + g^{*}_{\vec{k}}\hat{\sigma}_{-}\right)\left(\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}}\right)}_{H_{AF}}$$

For simplicity $\vec{\mu}_{21} = \vec{\mu}_{12} \Rightarrow g_{\vec{k}} = g^{*}_{\vec{k}} = g_{\vec{k}}$

Note: $\hat{H}_{AF}$ conserves excitation number, couples $|2,n\rangle \leftrightarrow |1,n+1\rangle$

Series of 2-level systems, one for each n

All 2-level systems are alike
Rabi problem!

Switch to Interaction Picture:

$$\left.\begin{aligned} \hat{H}_S \rightarrow \hat{H}_I &= e^{i\frac{\hat{H}_0}{\hbar}t}\,\hat{H}_{AF}\,e^{-i\frac{\hat{H}_0}{\hbar}t} \\ |\psi_S(t)\rangle \rightarrow |\psi_I(t)\rangle &= e^{i\frac{\hat{H}_0}{\hbar}t}\,|\psi_S(t)\rangle \end{aligned}\right\} \Rightarrow$$

Quantized Light – Matter Interactions

More about Single-mode Cavity QED

$$\hat{H} = \underbrace{\hbar\omega\hat{a}^{+}\hat{a} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar(g_{\vec{k}}\hat{\sigma}_{+} + g^{*}_{\vec{k}}\hat{\sigma}_{-})(\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})}_{H_{AF}}$$

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Can show

$$e^{i\omega\hat{a}^{+}\hat{a}t}\,\hat{a}\,e^{-i\omega\hat{a}^{+}\hat{a}t} = \hat{a}\,e^{-i\omega t}$$

$$e^{i\frac{\omega_{21}}{2}\hat{\sigma}_z t}\,\hat{\sigma}_{+}\,e^{-i\frac{\omega_{21}}{2}\hat{\sigma}_z t} = \hat{\sigma}_{+}\,e^{-i\omega_{21}t}$$

Quantized Light – Matter Interactions

More about Single-mode Cavity QED

$$\hat{H} = \underbrace{\hbar\omega\hat{a}^{+}\hat{a} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar(g_k\hat{\sigma}_{+} + g_k^{*}\hat{\sigma}_{-})(\hat{a}_k + \hat{a}_k^{+})}_{H_{AF}}$$

For simplicity $\vec{\mu}_{21} = \vec{\mu}_{12} \Rightarrow g_k = g_k^{*} = g_k$

Note: $\hat{H}_{AF}$ conserves excitation number, couples $|2,n\rangle \leftrightarrow |1,n+1\rangle$

Series of 2-level systems, one for each n

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$$\hat{H}_{AF} = \hbar g(\hat{\sigma}_{+}\hat{a}e^{i(\omega_{21}-\omega)t} + \hat{\sigma}_{+}\hat{a}^{+}e^{i(\omega_{21}+\omega)t} + \hat{\sigma}_{-}\hat{a}e^{-i(\omega_{21}+\omega)t} + \hat{\sigma}_{-}\hat{a}^{+}e^{-i(\omega_{21}-\omega)t})$$

RWA and resonant approximation

Jaynes-Cummings Hamiltonian

$$\boxed{\hat{H}_I = \hat{H}_{AF} = \hbar g(\hat{\sigma}_{+}\hat{a}e^{i\Delta t} + \hat{\sigma}_{-}\hat{a}^{+}e^{-i\Delta t})}$$

$\Delta = \omega_{21} - \omega$

Eigenstates of $\hat{H}_0 = \hat{H}_F + \hat{H}_A$

State	Energy
$\|2,n\rangle$	$\hbar\omega n + \frac{1}{2}\hbar\omega_{21}$
$\|1,n+1\rangle$	$\hbar\omega(n+1) - \frac{1}{2}\hbar\omega_{21}$

Quantized Light – Matter Interactions

$$\hat{H}_{AF} = \hbar g\left(\hat{\sigma}_+\hat{a}e^{i(\omega_{21}-\omega)t} + \hat{\sigma}_+\hat{a}^+e^{i(\omega_{21}+\omega)t} + \hat{\sigma}_-\hat{a}e^{-i(\omega_{21}+\omega)t} + \hat{\sigma}_-\hat{a}^+e^{-i(\omega_{21}-\omega)t}\right)$$

RWA and resonant approximation

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$\|2,n\rangle$	$\hbar\omega n + \frac{1}{2}\hbar\omega_{21}$
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Cavity QED version of the Rabi Problem

$$|\psi(0)\rangle = |2,n\rangle$$

$$|\psi(t)\rangle = C_{1,n+1}|1,n+1\rangle + C_{2,n}|2,n\rangle$$

Matrix elements

$$\langle 2,n|\hat{H}_{AF}|1,n+1\rangle = \hbar g\sqrt{n+1}\,e^{i\Delta t}$$

$$\langle 1,n+1|\hat{H}_{AF}|2,n\rangle = \hbar g\sqrt{n+1}\,e^{-i\Delta t}$$

Schrödinger Equation

$$i\hbar\frac{d}{dt}\begin{pmatrix} C_{1,n+1} \\ C_{2,n} \end{pmatrix} = \hbar g\sqrt{n+1}\begin{pmatrix} 0 & e^{-i\Delta t} \\ e^{i\Delta t} & 0 \end{pmatrix}\begin{pmatrix} C_{1,n+1} \\ C_{2,n} \end{pmatrix}$$

Quantized Light – Matter Interactions

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$$\boxed{\begin{aligned}\dot{C}_{1,n+1} &= -ig\sqrt{n+1}\,e^{-i\Delta t}\,C_{2,n} \\ \dot{C}_{2,n} &= -ig\sqrt{n+1}\,e^{i\Delta t}\,C_{1,n+1}\end{aligned}}$$

Substitute $C_{1,n+1} \to C_1\,,\quad C_{2,n} \to C_2 e^{i\Delta t}$

Looks exactly like Semiclassical Rabi problem

Solve for $C_1(0) = 0\,,\quad C_2(0) = 1$

$$C_{2,n}(t) = \left[\cos\left(\frac{\Omega_n t}{2}\right) - i\frac{\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]e^{i\Delta t/2}$$

$$C_{1,n+1}$$

Quantized Light – Matter Interactions

Cavity QED version of the Rabi Problem

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$$C_{1,n+1} = -i\,\frac{2g\sqrt{n+1}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)$$

Quantized Light – Matter Interactions

Cavity QED version of the Rabi Problem

$$|\psi(0)\rangle = |2,n\rangle$$

$$|\psi(t)\rangle = C_{1,n+1}\,|1,n+1\rangle + C_{2,n}\,|2,n\rangle$$

Matrix elements

$$\langle 2,n|\hat{H}_{AF}|1,n+1\rangle = \hbar g\sqrt{n+1}\,e^{i\Delta t}$$

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Schrödinger Equation

$$i\hbar\frac{d}{dt}\begin{pmatrix} C_{1,n+1} \\ C_{2,n}\end{pmatrix} =$$

$$\hbar g\sqrt{n+1}\begin{pmatrix} 0 & e^{-i\Delta t} \\ e^{i\Delta t} & 0\end{pmatrix}\begin{pmatrix} C_{1,n+1} \\ C_{2,n}\end{pmatrix}$$

$$\boxed{\begin{aligned} \dot{C}_{1,n+1} &= -ig\sqrt{n+1}\,e^{-i\Delta t}\,C_{2,n} \\ \dot{C}_{2,n} &= -ig\sqrt{n+1}\,e^{i\Delta t}\,C_{1,n+1}\end{aligned}}$$

Substitute $C_{1,n+1} \rightarrow C_1$, $C_{2,n} \rightarrow C_2 e^{i\Delta t}$

Looks exactly like Semiclassical Rabi problem

Solve for $C_1(0) = 0$, $C_2(0) = 1$

$$\boxed{\begin{aligned} C_{2,n}(t) &= \left[\cos\left(\frac{\Omega_n t}{2}\right) - i\frac{\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]e^{i\Delta t/2} \\ C_{1,n+1} &= -i\,\frac{2g\sqrt{n+1}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)e^{-i\Delta t/2} \\ \Omega_n &= \left(4g^2(n+1) + \Delta^2\right)^{1/2}\end{aligned}}$$

Quantized Light – Matter Interactions

$$\dot{c}_{1,n+1} = -ig\sqrt{n+1}\, e^{-i\Delta t} c_{2,n}$$

$$\dot{c}_{2,n} = -ig\sqrt{n+1}\, e^{i\Delta t} c_{1,n+1}$$

Substitute $c_{1,n+1} \rightarrow c_1$, $c_{2,n} \rightarrow c_2 e^{i\Delta t}$

Looks exactly like Semiclassical Rabi problem

Solve for $c_1(0) = 0$, $c_2(0) = 1$

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$$c_{1,n+1} = -i\,\frac{2g\sqrt{n+1}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right) e^{-i\Delta t/2}$$

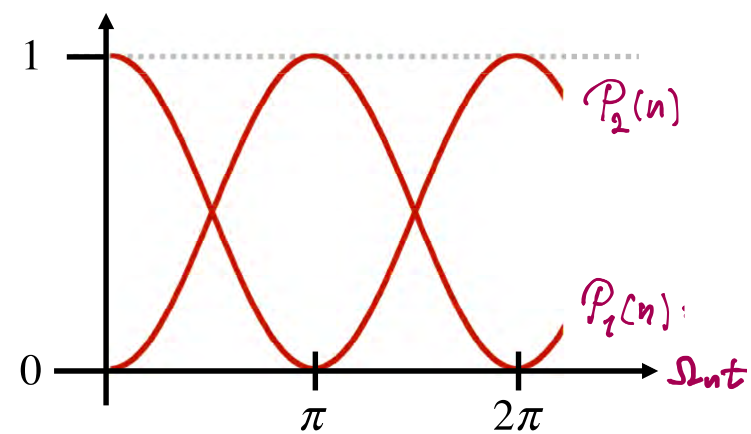
$$\Omega_n = \left(4g^2(n+1) + \Delta^2\right)^{1/2}$$

Rabi Oscillations

$$P_2(n) = \cos^2\left(\frac{\Omega_n t}{2}\right) + \left(\frac{\Delta}{\Omega_n}\right)^2 \sin^2\left(\frac{\Omega_n t}{2}\right)$$

$$P_1(n) = \frac{4g^2(n+1)}{\Omega_n^2}\sin^2\left(\frac{\Omega_n t}{2}\right)$$

Example: $\Delta = 0$



Quantized Light – Matter Interactions

Rabi Oscillations

$$P_2(n) = \cos^2\left(\frac{\Omega_n t}{2}\right) + \left(\frac{\Delta}{\Omega_n}\right)^2 \sin^2\left(\frac{\Omega_n t}{2}\right)$$

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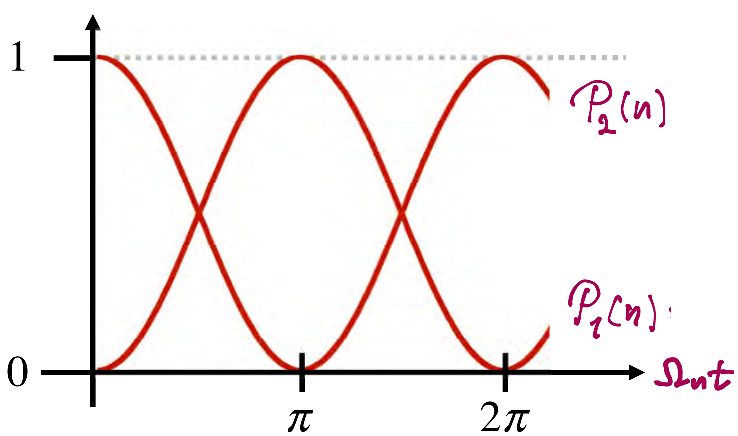

Vacuum Rabi Oscillations

If $|\psi(0)\rangle = |2,0\rangle$ ⇨ no photons in field

yet $|2,0\rangle$ evolves into $|1,1\rangle$

Uniquely QED phenomenon!

Asymmetry $\begin{cases} |2,n=0\rangle \longrightarrow |1,n=1\rangle \\ |1,n=0\rangle \longrightarrow |1,n=1\rangle \end{cases}$

holds germ of **Spontaneous Decay**

Quantized Light – Matter Interactions

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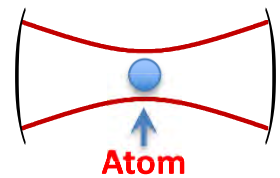
Next: More Cavity QED



2-level atom

Single cavity mode

What happens with a Coherent State in the Cavity mode?

(Quantum-Classical correspondence)

Initial atom-field state:

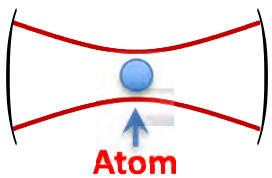
$$|\psi(0)\rangle = \underbrace{|1\rangle}_{\text{atom}} \otimes \underbrace{|\alpha\rangle}_{\text{field}} = \sum_n C_n |1,n\rangle, \quad C_n = e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}}$$

From Rabi solutions: $\Delta = 0$ ⇨

$$C_{1,n} = \cos\left(\frac{\Omega_n t}{2}\right), \quad C_{2,n-1} = -i\sin\left(\frac{\Omega_n t}{2}\right)$$

Quantized Light – Matter Interactions

Today: More Cavity QED



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Therefore

uncoupled

$$\boxed{|\psi(t)\rangle = C_0 |1,0\rangle + \sum_{n=1}^{\infty} C_n \left[\cos\left(\frac{\Omega_n t}{2}\right)|1,n\rangle - i \sin\left(\frac{\Omega_n t}{2}\right)|2,n-1\rangle\right]}$$

Consider the Atomic Excited State Population

$$P_2(t) = \sum_{n=0}^{\infty} P_{2,n} = \sum_{n=0}^{\infty} |\langle 2,n|\psi(t)\rangle|^2$$

$$= \sum_{n=0}^{\infty} |C_n|^2 \sin^2\left(\frac{\Omega_n t}{2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} \sin^2\left(\frac{\Omega_n t}{2}\right)$$

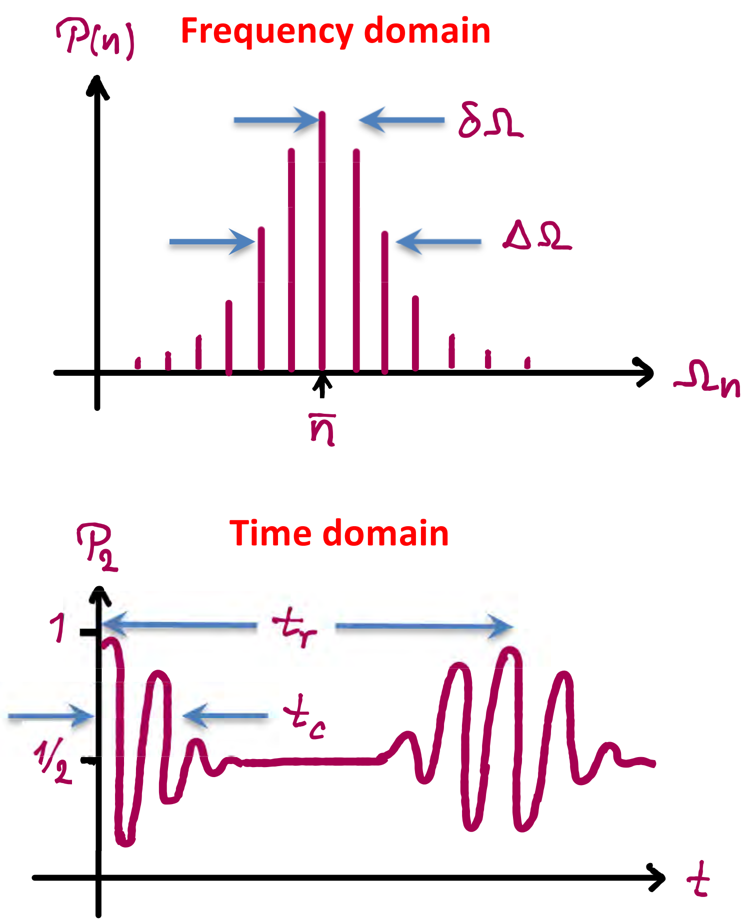
Use $|\alpha|^2 = \bar{n}$ **and** $\Omega_n = 2g\sqrt{n}$

$$\boxed{P_2(t) = \sum_{n=0}^{\infty} \frac{(\bar{n})^n e^{-\bar{n}}}{n!} \sin^2(g\sqrt{n}\, t)}$$

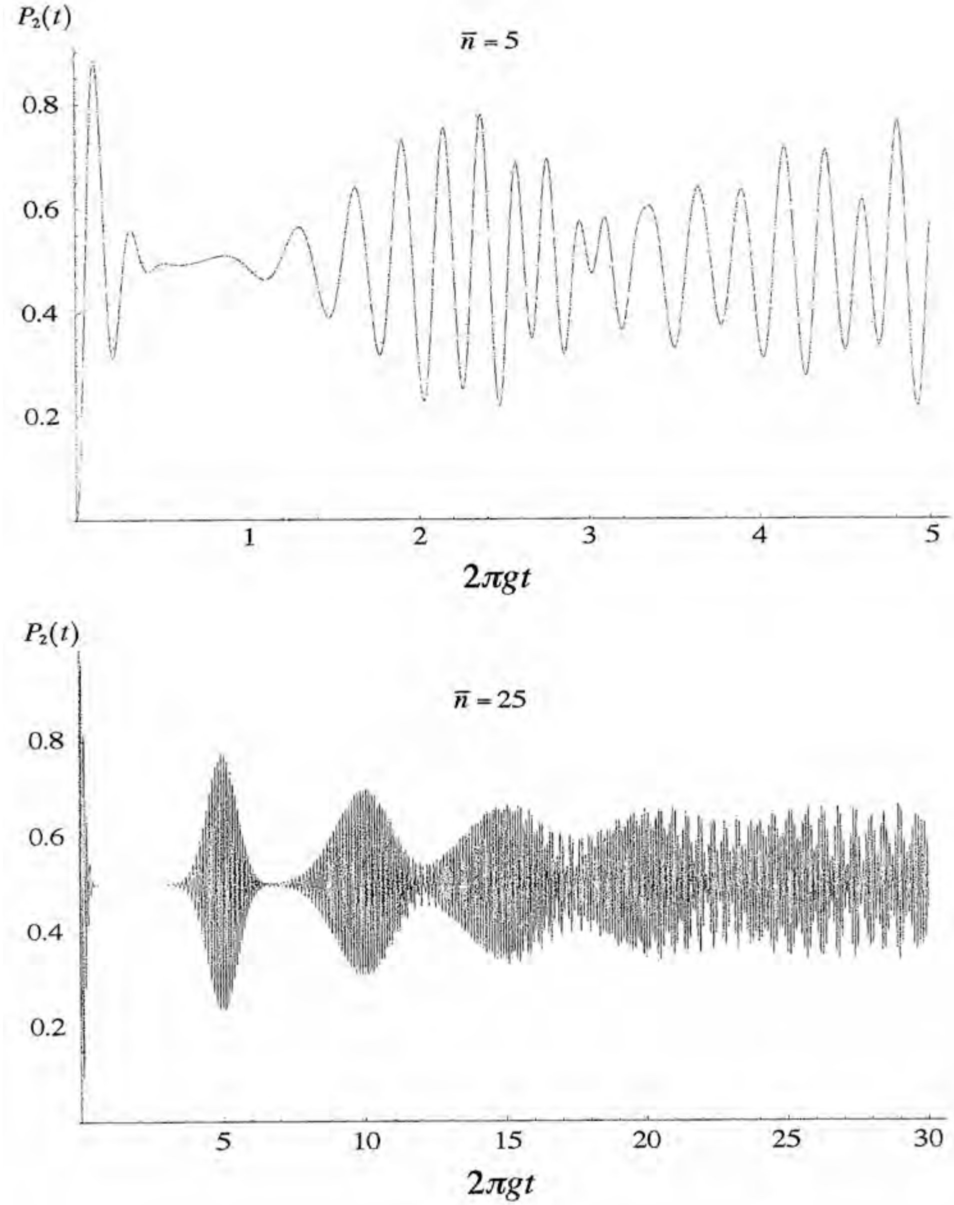
Quantized Light – Matter Interactions

– **Poisson weighted average of sinusoids**
– **Sinusoids gradually dephase over time**

Collapse of oscillation amplitude



Numerical examples



Quantized Light – Matter Interactions

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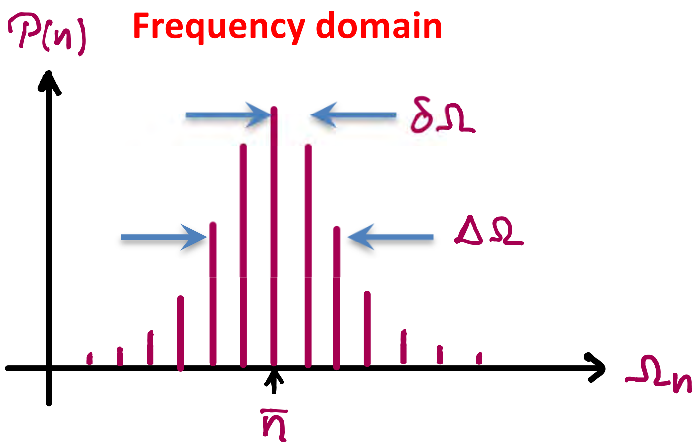

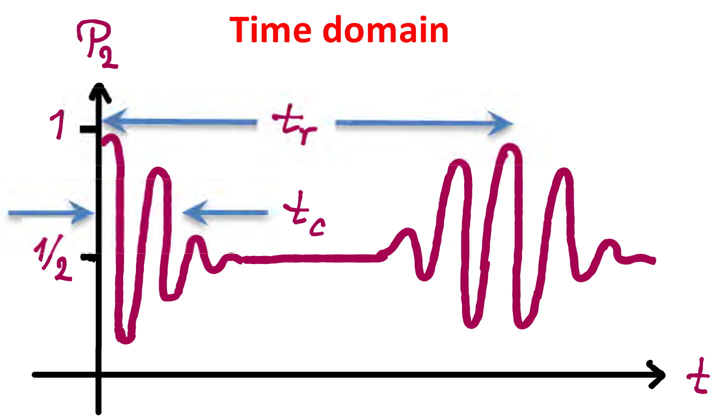

Use $\Delta n = \sqrt{\bar{n}}$ ⇨ $\Delta\Omega \sim \Delta\Omega_{\bar{n}+\sqrt{\bar{n}}} - \Delta\Omega_{\bar{n}-\sqrt{\bar{n}}}$

$$\boxed{t_c = \frac{1}{\Delta\Omega} \sim \frac{1}{2g\sqrt{\bar{n}+\sqrt{\bar{n}}} - 2g\sqrt{\bar{n}-\sqrt{\bar{n}}}} \sim \frac{1}{2g}}$$

for $\bar{n} \gg \sqrt{\bar{n}}$

Rephasing: when $(\Omega_{\bar{n}} - \Omega_{\bar{n}-1})\, t_r \simeq 2\pi m$

Similar arguments -> Revival time

$$\boxed{t_r \sim \frac{2\pi}{\delta\Omega} \sim \frac{2\pi\sqrt{\bar{n}}}{g}}$$

Quantized Light – Matter Interactions

Use $\Delta n = \sqrt{\bar{n}} \Rightarrow \Delta\Omega \sim \Delta\Omega_{\bar{n}+\sqrt{\bar{n}}} - \Delta\Omega_{\bar{n}-\sqrt{\bar{n}}}$

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Collapse & Revival Dynamics

Pure Quantum Phenomenon

("graininess" of photons)

Classical limit $\left\{ \begin{array}{l} g \to 0 \\ \mathcal{E}_k \to 0 \\ \frac{\Delta\Omega}{\Omega_{\bar{n}}} \to 0 \end{array} \right.$ $\Rightarrow$ $\begin{array}{l} \bar{n} \to \infty \\ t_c \to \infty \end{array}$ $\Rightarrow$ $\Omega_{\bar{n}} \neq 0$ **well defined**

$$\boxed{\Omega_{\bar{n}} = 2g\sqrt{\bar{n}} = \frac{\vec{\mu}_{21} \cdot 2\vec{\varepsilon}_k \mathcal{E}_k \sqrt{\bar{n}}}{\hbar} = \frac{\vec{\mu}_{21} \cdot \vec{E}}{\hbar}}$$

Classical Rabi frequency

mean field $\langle \alpha(t) | \hat{\vec{E}} | \alpha(t) \rangle$