Quantized Light – Matter Interactions

General Problem:

Quantum Atom ⟷ Quantized Modes Of EM Field

Electric Dipole Interaction

Starting Point: System Hamiltonian

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} \qquad \textbf{(1)}$$

$$\hat{H}_F = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \left(\hat{a}^{+}_{\vec{k}}\hat{a}_{\vec{k}} + \tfrac{1}{2}\right) \qquad \text{Field}$$

$$\hat{H}_A = \sum_i E_i |i\rangle\langle i| = \sum_i E_i \hat{\sigma}_{ii} \qquad \text{Atom}$$

$$\hat{H}_{AF} = -\hat{\vec{\mu}}\cdot\hat{\vec{E}}(\vec{r},t) \qquad \text{ED interaction}$$

E_i, $|i\rangle$: energies, energy levels of the atom

Dipole Operator:

$$\textbf{(2)} \qquad \hat{\vec{\mu}} = \sum_{i,j} \vec{\mu}_{ij} |i\rangle\langle j| = \sum_{i,j} \vec{\mu}_{ij}\hat{\sigma}_{ij}$$

Field Operator:

$$\vec{E}(\vec{r},t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + H.c., \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar\omega_{\vec{k}}}{2\varepsilon_0 V}}$$

2 polarization modes implicit

Pin down atom where $u_{\vec{k}}(\vec{r}) = 1$

– anywhere if $u_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$

– if $u_{\vec{k}}(\vec{r}) = \sin(kz)$ then where $\sin(kz) = 1$

$$\textbf{(3)} \qquad \hat{\vec{E}}(\vec{r},t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \left(\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}}\right)$$

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Dipole Operator:

(2) $$\boxed{\hat{\vec{\mu}} = \sum_{i,j} \vec{\mu}_{ij} |i\rangle\langle j| = \sum_{i,j} \vec{\mu}_{ij}\hat{\sigma}_{ij}}$$

Field Operator:

$$\vec{E}(\vec{r},t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + H.C., \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar\omega_{\vec{k}}}{2\varepsilon_0 V}}$$

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– if $u_{\vec{k}}(\vec{r}) = \sin(kz)$ then where $\sin(kz) = 1$

(3) $$\boxed{\hat{\vec{E}}(\vec{r},t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})}$$

Combining (2) & (3):

$$\hat{H}_{AF} = \sum_{ij}\sum_{\vec{k}} -\vec{\mu}_{ij}\cdot\vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})$$

$$= \sum_{ij}\sum_{\vec{k}} \hbar g^{(ij)}_{\vec{k}} \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})$$

where $\hbar g^{(ij)}_{\vec{k}} = \vec{\mu}_{ij}\cdot\vec{\varepsilon}_{\vec{k}}\mathcal{E}_{\vec{k}}$

Rabi Freq., same sign convention as when we first looked at the Rabi problem

2-level atom ⇨ $(i,j) = (1,2)$:

$$\boxed{\hat{H}_{AF} = \hbar \sum_{\vec{k}} (g_{\vec{k}} \hat{\sigma}_{21} + g^{*}_{\vec{k}} \hat{\sigma}_{12})(\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})}$$

Define:

$$\left.\begin{aligned} \hat{\sigma}_{+} &= \hat{\sigma}_{21} = |2\rangle\langle 1| \\ \hat{\sigma}_{-} &= \hat{\sigma}_{12} = |1\rangle\langle 2| \\ \hat{\sigma}_{z} &= \hat{\sigma}_{22} - \hat{\sigma}_{11} = |2\rangle\langle 2| - |1\rangle\langle 1| \end{aligned}\right\} \Rightarrow$$

Pauli matrices

$$\hat{\sigma}_{x} = \frac{1}{2}(\hat{\sigma}_{+} + \hat{\sigma}_{-})$$

$$\hat{\sigma}_{y} = \frac{1}{2i}(\hat{\sigma}_{+} - \hat{\sigma}_{-})$$

$$\hat{\sigma}_{z}$$

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Combining (2) & (3):

$$\boxed{\begin{aligned}\hat H_{AF} &= \sum_{ij}\sum_{\vec k} -\vec\mu_{ij}\cdot\vec\varepsilon_{\vec k}\,\mathcal{E}_{\vec k}\,\hat\sigma_{ij}\left(\hat a_{\vec k}+\hat a^{+}_{\vec k}\right)\\ &= \sum_{ij}\sum_{\vec k}\hbar g^{(ij)}_{\vec k}\,\hat\sigma_{ij}\left(\hat a_{\vec k}+\hat a^{+}_{\vec k}\right)\\ &\text{where}\quad \hbar g^{(ij)}_{\vec k} = \vec\mu_{ij}\cdot\vec\varepsilon_{\vec k}\,\mathcal{E}_{\vec k}\end{aligned}}$$

Rabi Freq., same sign convention as when we first looked at the Rabi problem

2-level atom ⇨ $(i,j)=(1,2)$ **:**

$$\boxed{\hat H_{AF} = \hbar\sum_{\vec k}\left(g_{\vec k}\hat\sigma_{21}+g^{*}_{\vec k}\hat\sigma_{12}\right)\left(\hat a_{\vec k}+\hat a^{+}_{\vec k}\right)}$$

Define:

$\hat\sigma_+ = \hat\sigma_{21} = |2\rangle\langle 1|$

$\hat\sigma_- = \hat\sigma_{12} = |1\rangle\langle 2|$

$\hat\sigma_z = \hat\sigma_{22}-\hat\sigma_{11} = |2\rangle\langle 2| - |1\rangle\langle 1|$

⇨ **Pauli matrices**

$\hat\sigma_x = \frac{1}{2}\left(\hat\sigma_+ + \hat\sigma_-\right)$

$\hat\sigma_y = \frac{1}{2i}\left(\hat\sigma_+ - \hat\sigma_-\right)$

$\hat\sigma_z$

With this notation **(4)**

$$\hat H_{AF} = \hbar\sum_{\vec k}\left(g_{\vec k}\hat\sigma_+\hat a_{\vec k} + \cancel{g_{\vec k}\hat\sigma_+\hat a^{+}_{\vec k}} + \cancel{g_{\vec k}\hat\sigma_-\hat a_{\vec k}} + g_{\vec k}\hat\sigma_-\hat a^{+}_{\vec k}\right)$$

Energy conservation?

$$\hat H_{AF} = \hbar\sum_{\vec k}\left(g_{\vec k}\hat\sigma_+\hat a_{\vec k} + g_{\vec k}\hat\sigma_-\hat a^{+}_{\vec k}\right)$$

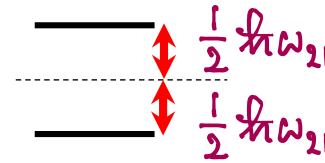
Putting it all together

$$\boxed{\begin{aligned}&\hat H = \hat H_F + \hat H_A + \hat H_{AF} = \qquad\qquad \textbf{(5)}\\ &\sum_{\vec k}\hbar\omega_{\vec k}\hat a^{+}_{\vec k}\hat a_{\vec k} + \frac{1}{2}\hbar\omega_{21}\hat\sigma_z + \sum_{\vec k}\hbar\left(g_{\vec k}\hat\sigma_+\hat a_{\vec k} + g_{\vec k}\hat\sigma_-\hat a^{+}_{\vec k}\right)\end{aligned}}$$

We changed the zero point for energy by subtracting

$\sum_{\vec k}\frac{1}{2}\hbar\omega_{\vec k}$ and $\frac{1}{2}(E_2-E_1)$

field atom



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With this notation

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + \cancel{g_{\vec{k}} \hat{\sigma}_+ \hat{a}^+_{\vec{k}}} + \cancel{g_{\vec{k}} \hat{\sigma}_- \hat{a}_{\vec{k}}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right) \quad (4)$$

Energy conservation?

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right)$$

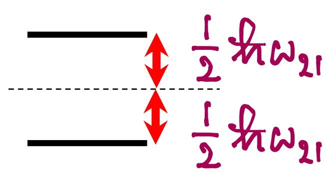
Putting it all together

$$\boxed{\begin{aligned} \hat{H} &= \hat{H}_F + \hat{H}_A + \hat{H}_{AF} = \\ &\sum_{\vec{k}} \hbar\omega_{\vec{k}} \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_z + \sum_{\vec{k}} \hbar \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right) \end{aligned}} \quad (5)$$

We changed the zero point for energy by subtracting

$\sum_{\vec{k}} \frac{1}{2}\hbar\omega_{\vec{k}}$ and $\frac{1}{2}(E_2 - E_1)$

field **atom**



Foundational result for remainder of course

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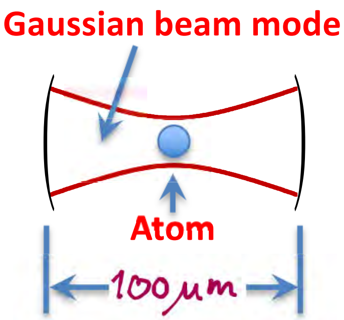
Interaction with Single-mode Fields

Good approx. in small, high-Q Cavity



$$c/2L \gg A_{21}$$

$$|g_{\vec{k}}| \gg A_{21}, \gamma$$

Single-mode (Jaynes-Cummings) Hamiltonian

$$\hat{H} = \underbrace{\hbar\omega\,\hat{a}^{+}\hat{a} + \tfrac{1}{2}\hbar\omega_{21}\hat{\sigma}_{z}}_{H_0} + \underbrace{\hbar g\,(\hat{\sigma}_{+} + \hat{\sigma}_{-})(\hat{a} + \hat{a}^{+})}_{H_{AF}}$$

Paradigm for Qubit coupled to QHO

- Atom in high-Q Cavity *)
- Quantum dot in high-Q Cavity
- Rydberg atom in superconducting μw Cavity
- Superconducting qubit in superconducting μw Cavity
- Superconducting qubit in superconducting μw stripline Cavity (circuit QED)
- Trapped ion with quantized COM motion *)

*) Nobel Prize in Physics 2012

Important Change in Notation

For the rest of this course we change indices 1 to *g* (ground state) and 2 to *e* (excited state). A *g* inside a ket refers to a state, a *g* elsewhere is a Rabi frequency. This is needed for clarity.

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Paradigm for Qubit coupled to QHO

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Important Change in Notation

For the rest of this course we change indices 1 to *g* (ground state) and 2 to *e* (excited state). A *g* in a ket or a subscript refers to a state, a *g* elsewhere is a Rabi frequency.

More about Single-mode Cavity QED

$$\hat{H} = \underbrace{\hbar\omega\hat{a}^{+}\hat{a} + \tfrac{1}{2}\hbar\omega_{21}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar(g_{\vec{k}}\hat{\sigma}_{+} + g_{\vec{k}}^{*}\hat{\sigma}_{-})(\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^{+})}_{H_{AF}}$$

For simplicity $\vec{\mu}_{21} = \vec{\mu}_{12} \Rightarrow g_{\vec{k}} = g_{\vec{k}}^{*} = g_{\vec{k}}$

Note: $\hat{H}_{AF}$ conserves excitation number, couples $|e,n\rangle \leftrightarrow |g,n+1\rangle$

$\Downarrow$

Series of 2-level systems, one for each n

All 2-level systems are alike
Rabi problem!

Switch to Interaction Picture:

Sakurai page 318-319

$$\left.\begin{aligned} \hat{H}_S &\to \hat{H}_I = e^{i\frac{\hat{H}_0}{\hbar}t}\hat{H}_{AF}e^{-i\frac{\hat{H}_0}{\hbar}t} \\ |\psi_S(t)\rangle &\to |\psi_I(t)\rangle = e^{i\frac{\hat{H}_0}{\hbar}t}|\psi_S(t)\rangle \end{aligned}\right\} \Rightarrow$$

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More about Single-mode Cavity QED

$$\hat{H} = \underbrace{\hbar\omega\,\hat{a}^+\hat{a} + \tfrac{1}{2}\hbar\omega_{21}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar\left(g_{\vec{k}}\hat{\sigma}_+ + g^*_{\vec{k}}\hat{\sigma}_-\right)\left(\hat{a}_{\vec{k}} + \hat{a}^+_{\vec{k}}\right)}_{H_{AF}}$$

For simplicity $\vec{\mu}_{21} = \vec{\mu}_{12}$ ⇨ $g_{\vec{k}} = g^*_{\vec{k}} = g_{\vec{k}}$

Note: $\hat{H}_{AF}$ **conserves excitation number,**
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⇩

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Can show

$$e^{i\omega\hat{a}^+\hat{a}t}\,\hat{a}\,e^{-i\omega\hat{a}^+\hat{a}t} = \hat{a}\,e^{-i\omega t}$$

$$e^{i\frac{\omega_{21}}{2}\hat{\sigma}_z t}\,\hat{\sigma}_+\,e^{-i\frac{\omega_{21}}{2}\hat{\sigma}_z t} = \hat{\sigma}_+\,e^{-i\omega_{21}t}$$

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More about Single-mode Cavity QED

$$\hat{H} = \underbrace{\hbar\omega\,\hat{a}^+\hat{a} + \tfrac{1}{2}\hbar\omega_{21}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar(g_k\hat{\sigma}_+ + g_k^*\hat{\sigma}_-)(\hat{a}_k + \hat{a}_k^+)}_{H_{AF}}$$

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Note: $\hat{H}_{AF}$ **conserves excitation number, couples** $|e,n\rangle \leftrightarrow |g,n+1\rangle$

$\Downarrow$

Series of 2-level systems, one for each n

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Switch to Interaction Picture:

$$\hat{H}_S \rightarrow \hat{H}_I = e^{i\frac{\hat{H}_0}{\hbar}t}\hat{H}_{AF}e^{-i\frac{\hat{H}_0}{\hbar}t}$$

$$|\psi_S(t)\rangle \rightarrow |\psi_I(t)\rangle = e^{i\frac{\hat{H}_0}{\hbar}t}|\psi_S(t)\rangle$$

$\Rightarrow$

$$\hat{H}_{AF} = \hbar g\left(\hat{\sigma}_+\hat{a}e^{i(\omega_{21}-\omega)t} + \hat{\sigma}_+\hat{a}^+e^{i(\omega_{21}+\omega)t} + \hat{\sigma}_-\hat{a}e^{-i(\omega_{21}+\omega)t} + \hat{\sigma}_-\hat{a}^+e^{-i(\omega_{21}-\omega)t}\right)$$

RWA and resonant approximation

$\Downarrow$

Jaynes-Cummings Hamiltonian

$$\boxed{\hat{H}_I = \hat{H}_{AF} = \hbar g\left(\hat{\sigma}_+\hat{a}e^{i\Delta t} + \hat{\sigma}_-\hat{a}^+e^{-i\Delta t}\right)}$$

$\Delta = \omega_{21} - \omega$

Eigenstates of $\hat{H}_0 = \hat{H}_F + \hat{H}_A$

State	Energy
$\lvert e,n\rangle$	$\hbar\omega n + \frac{1}{2}\hbar\omega_{21}$
$\lvert g,n+1\rangle$	$\hbar\omega(n+1) - \frac{1}{2}\hbar\omega_{21}$

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$$\hat{H}_{AF} = \hbar g \left(\hat{\sigma}_+ \hat{a} e^{i(\omega_{21}-\omega)t} + \hat{\sigma}_+ \hat{a}^+ e^{i(\omega_{21}+\omega)t} + \hat{\sigma}_- \hat{a} e^{-i(\omega_{21}+\omega)t} + \hat{\sigma}_- \hat{a}^+ e^{-i(\omega_{21}-\omega)t} \right)$$

RWA and resonant approximation

Jaynes-Cummings Hamiltonian

$$\boxed{\hat{H}_I = \hat{H}_{AF} = \hbar g \left(\hat{\sigma}_+ \hat{a} e^{i\Delta t} + \hat{\sigma}_- \hat{a}^+ e^{-i\Delta t} \right)}$$

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Cavity QED version of the Rabi Problem

$$\lvert\psi(0)\rangle = \lvert e,n\rangle$$

$$\lvert\psi(t)\rangle = C_{g,n+1}\lvert g,n+1\rangle + C_{e,n}\lvert e,n\rangle$$

Matrix elements

$$\langle e,n\rvert \hat{H}_{AF} \lvert g,n+1\rangle = \hbar g\sqrt{n+1}\, e^{i\Delta t}$$

$$\langle g,n+1\rvert \hat{H}_{AF} \lvert e,n\rangle = \hbar g\sqrt{n+1}\, e^{-i\Delta t}$$

Schrödinger Equation

$$i\hbar \frac{d}{dt}\begin{pmatrix} C_{g,n+1} \\ C_{e,n} \end{pmatrix} = \hbar g\sqrt{n+1}\begin{pmatrix} 0 & e^{-i\Delta t} \\ e^{i\Delta t} & 0 \end{pmatrix}\begin{pmatrix} C_{g,n+1} \\ C_{e,n} \end{pmatrix}$$

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Cavity QED version of the Rabi Problem

$$|\psi(0)\rangle = |e,n\rangle$$

$$|\psi(t)\rangle = C_{g,n+1}|g,n+1\rangle + C_{e,n}|e,n\rangle$$

Matrix elements

$$\langle e,n|\hat{H}_{AF}|g,n+1\rangle = \hbar g\sqrt{n+1}\,e^{i\Delta t}$$

$$\langle g,n+1|\hat{H}_{AF}|e,n\rangle = \hbar g\sqrt{n+1}\,e^{-i\Delta t}$$

Schrödinger Equation

$$i\hbar\frac{d}{dt}\begin{pmatrix} C_{g,n+1} \\ C_{e,n}\end{pmatrix} = \hbar g\sqrt{n+1}\begin{pmatrix} 0 & e^{-i\Delta t} \\ e^{i\Delta t} & 0\end{pmatrix}\begin{pmatrix} C_{g,n+1} \\ C_{e,n}\end{pmatrix}$$

$$\boxed{\begin{aligned}\dot{C}_{g,n+1} &= -ig\sqrt{n+1}\,e^{-i\Delta t}\,C_{e,n} \\ \dot{C}_{e,n} &= -ig\sqrt{n+1}\,e^{i\Delta t}\,C_{g,n+1}\end{aligned}}$$

Substitute $C_{g,n+1} \to C_1$, $C_{e,n} \to C_2 e^{i\Delta t}$

Looks exactly like Semiclassical Rabi problem

Solve for $C_e(0) = 0$, $C_n(0) = 1$

$$\boxed{\begin{aligned}C_{e,n}(t) &= \left[\cos\left(\frac{\Omega_n t}{2}\right) - i\frac{\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]e^{i\Delta t/2} \\ C_{g,n+1} &= -i\,\frac{2g\sqrt{n+1}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)e^{-i\Delta t/2} \\ \Omega_n &= \left(4g^2(n+1) + \Delta^2\right)^{1/2}\end{aligned}}$$

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$$\dot{c}_{g,n+1} = -ig\sqrt{n+1}\, e^{-i\Delta t}\, c_{e,n}$$

$$\dot{c}_{e,n} = -ig\sqrt{n+1}\, e^{i\Delta t}\, c_{g,n+1}$$

Substitute $c_{g,n+1} \rightarrow c_1\ ,\ c_{e,n} \rightarrow c_2 e^{i\Delta t}$

Looks exactly like Semiclassical Rabi problem

Solve for $c_e(0) = 0\ ,\ c_g(0) = 1$

$$c_{e,n}(t) = \left[\cos\left(\frac{\Omega_n t}{2}\right) - i\frac{\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right] e^{i\Delta t/2}$$

$$c_{g,n+1} = -i\,\frac{2g\sqrt{n+1}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right) e^{-i\Delta t/2}$$

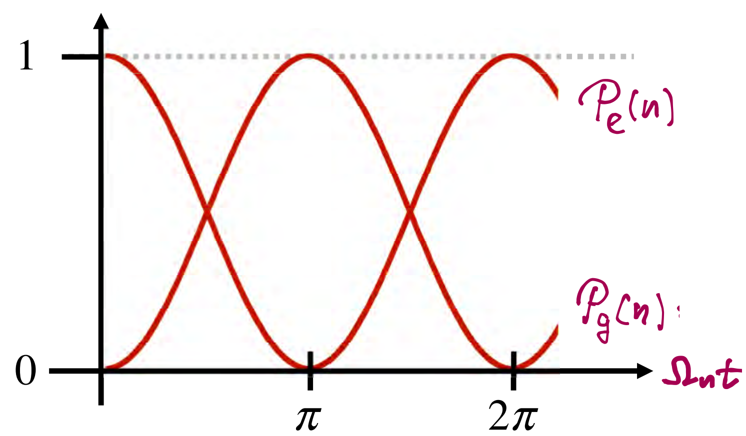
$$\Omega_n = \left(4g^2(n+1) + \Delta^2\right)^{1/2}$$

Rabi Oscillations

$$P_e(n) = \cos^2\left(\frac{\Omega_n t}{2}\right) + \left(\frac{\Delta}{\Omega_n}\right)^2 \sin^2\left(\frac{\Omega_n t}{2}\right)$$

$$P_g(n) = \frac{4g^2(n+1)}{\Omega_n^2}\sin^2\left(\frac{\Omega_n t}{2}\right)$$

Example: $\Delta = 0$



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Rabi Oscillations

$$P_e(n) = \cos^2\left(\frac{\Omega_n t}{2}\right) + \left(\frac{\Delta}{\Omega_n}\right)^2 \sin^2\left(\frac{\Omega_n t}{2}\right)$$

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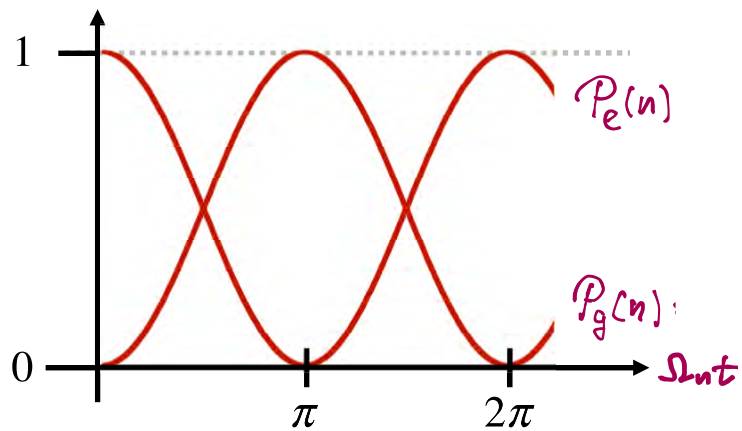

Vacuum Rabi Oscillations

If $|\psi(0)\rangle = |e,0\rangle$ ⇨ no photons in field

yet $|e,0\rangle$ evolves into $|1,1\rangle$

Uniquely QED phenomenon!

Asymmetry
$$\begin{cases} |e, n=0\rangle \rightarrow |g, n=1\rangle \\ |g, n=0\rangle \rightarrow |g, n=0\rangle \end{cases}$$

Schrödinger evolution

holds germ **of Spontaneous Decay**

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Vacuum Rabi Oscillations

If $|\psi(0)\rangle = |e,0\rangle$ ⇨ no photons in field

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Uniquely QED phenomenon!

Asymmetry $\begin{cases} |e, n=0\rangle \longrightarrow |g, n=1\rangle \\ |g, n=0\rangle \longrightarrow |g, n=0\rangle \end{cases}$

Schrödinger evolution

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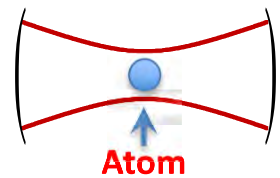
Next: More Cavity QED



2-level atom

Single cavity mode

What happens with a Coherent State in the Cavity mode?

(Quantum-Classical correspondence)

Initial atom-field state:

$$|\psi(0)\rangle = \underbrace{|g\rangle}_{\text{atom}} \otimes \underbrace{|\alpha\rangle}_{\text{field}} = \sum_n C_n |g,n\rangle, \quad C_n = e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}}$$

From Rabi solutions: $\Delta = 0$ ⇨

$$C_{g,n} = \cos\left(\frac{\Omega_n t}{2}\right), \quad C_{e,n-1} = -i \sin\left(\frac{\Omega_n t}{2}\right)$$

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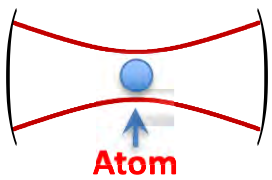
Today: More Cavity QED



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From Rabi solutions:

$$C_{g,n} = \cos\left(\frac{\Omega_n t}{2}\right), \quad C_{e,n-1} = -i\sin\left(\frac{\Omega_n t}{2}\right)$$

Therefore

$$\boxed{|\psi(t)\rangle = C_0 \underbrace{|g,0\rangle}_{\text{uncoupled}} + \sum_{n=1}^{\infty} C_n\left[\cos\left(\frac{\Omega_n t}{2}\right)|g,n\rangle - i\sin\left(\frac{\Omega_n t}{2}\right)|e,n-1\rangle\right]}$$

Consider the Atomic Excited State Population

$$\mathcal{P}_e(t) = \sum_{n=0}^{\infty} \mathcal{P}_{e,n} = \sum_{n=0}^{\infty} |\langle e,n|\psi(t)\rangle|^2$$

$$= \sum_{n=0}^{\infty} |C_n|^2 \sin^2\left(\frac{\Omega_n t}{2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} \sin^2\left(\frac{\Omega_n t}{2}\right)$$

Use $|\alpha|^2 = \bar{n}$ and $\Omega_n = 2g\sqrt{n}$

$$\boxed{\mathcal{P}_e(t) = \sum_{n=0}^{\infty} \frac{(\bar{n})^n e^{-\bar{n}}}{n!} \sin^2(g\sqrt{n}\,t)}$$

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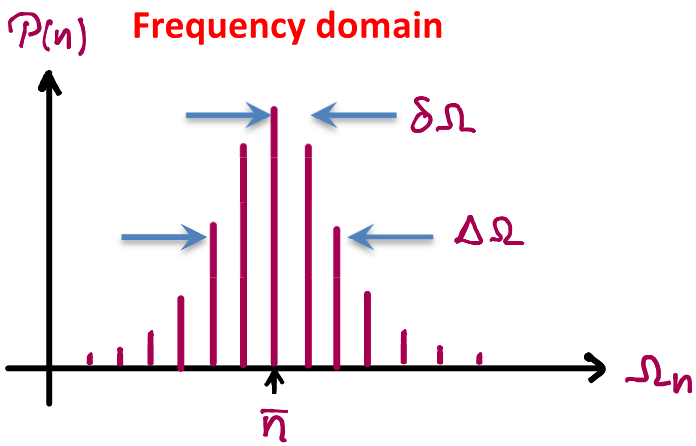
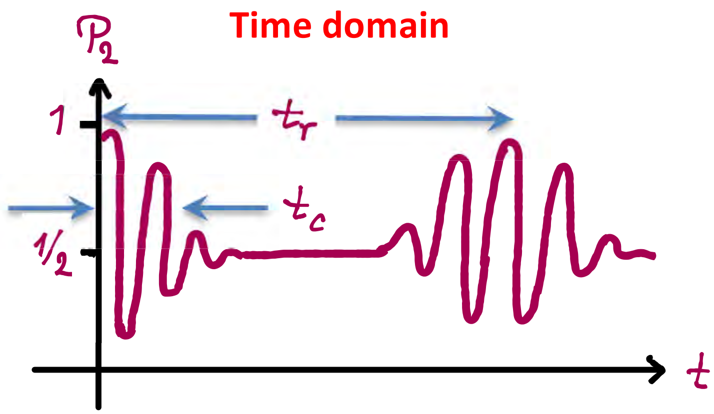
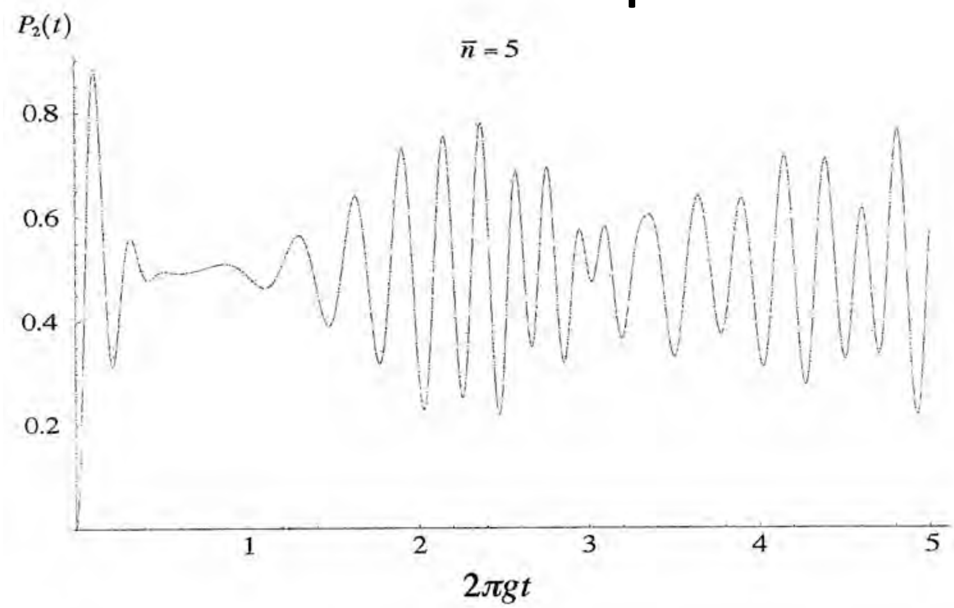
– Poisson weighted average of sinusoids
– Sinusoids gradually dephase over time

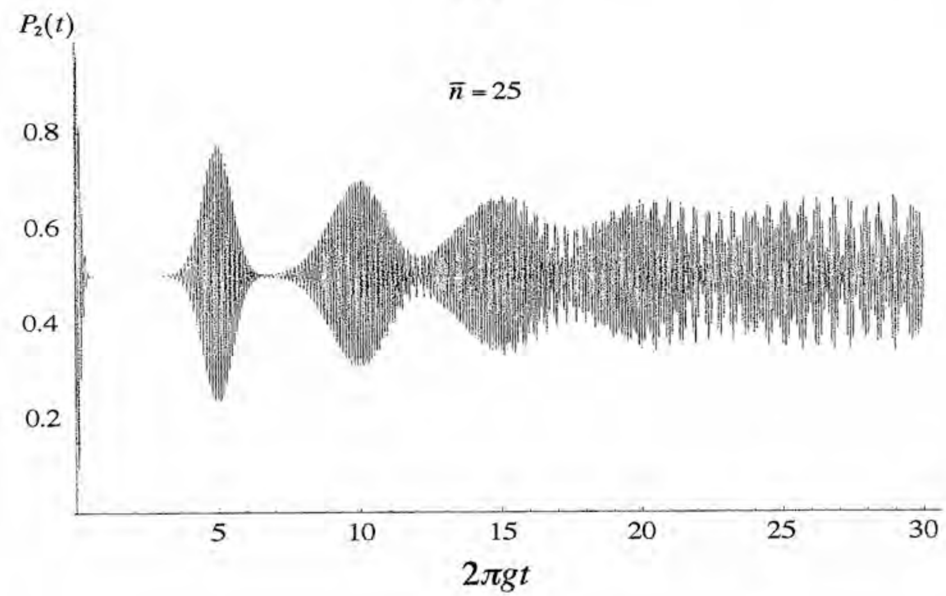
Collapse of oscillation amplitude





Numerical examples

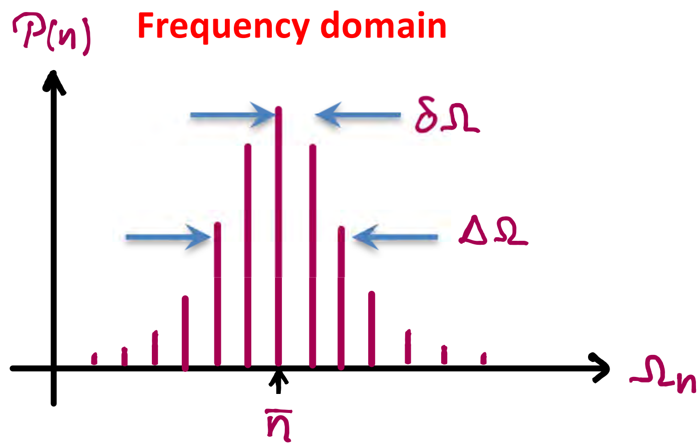


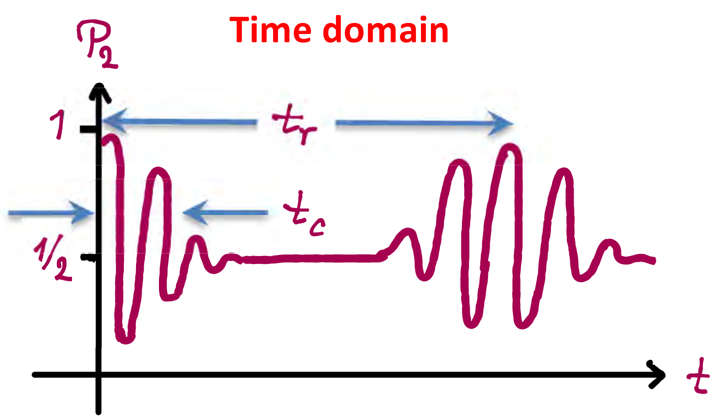


Quantized Light – Matter Interactions

– Poisson weighted average of sinusoids
– Sinusoids gradually dephase over time

Collapse of oscillation amplitude





Use $\Delta n = \sqrt{\bar{n}} \Rightarrow \Delta\Omega \sim \Delta\Omega_{\bar{n}+\sqrt{\bar{n}}} - \Delta\Omega_{\bar{n}-\sqrt{\bar{n}}}$

$$\boxed{t_c = \frac{1}{\Delta\Omega} \sim \frac{1}{2g\sqrt{\bar{n}+\sqrt{\bar{n}}} - 2g\sqrt{\bar{n}-\sqrt{\bar{n}}}} \sim \frac{1}{2g}}$$

for $\bar{n} \gg \sqrt{\bar{n}}$

Rephasing: **when** $(\Omega_{\bar{n}} - \Omega_{\bar{n}-1})\, t_r \simeq 2\pi m$

Similar arguments -> Revival time

$$\boxed{t_r \sim \frac{2\pi}{\delta\Omega} \sim \frac{2\pi\sqrt{\bar{n}}}{g}}$$

Quantized Light – Matter Interactions

Use $\Delta n = \sqrt{\bar{n}}$ ➡ $\Delta\Omega \sim \Delta\Omega_{\bar{n}+\sqrt{\bar{n}}} - \Delta\Omega_{\bar{n}-\sqrt{\bar{n}}}$

$$\boxed{t_c = \frac{1}{\Delta\Omega} \sim \frac{1}{2g\sqrt{\bar{n}+\sqrt{\bar{n}}} - 2g\sqrt{\bar{n}-\sqrt{\bar{n}}}} \sim \frac{1}{2g}}$$

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Similar arguments -> Revival time

$$\boxed{t_r \sim \frac{2\pi}{\delta\Omega} \sim \frac{2\pi\sqrt{\bar{n}}}{g}}$$

Collapse & Revival Dynamics

Pure Quantum Phenomenon

("graininess" of photons)

Classical limit:

Let the mode volume $V \to \infty$ and thus both $\mathcal{E}_k \to 0$ and $g \to 0$. Then let $\bar{n} \to \infty$ such that the total field strength remains constant.

Then $t_c \to \infty$, $\frac{\Delta\Omega}{\Omega_{\bar{n}}} \to 0$ and $\Omega_{\bar{n}}$ is well defined

$$\boxed{\Omega_{\bar{n}} = 2g\sqrt{\bar{n}} = \frac{\vec{\mu}_{eg}\cdot 2\vec{\varepsilon}_k \mathcal{E}_k \sqrt{\bar{n}}}{\hbar} = \frac{\vec{\mu}_{eg}\cdot\vec{E}}{\hbar}}$$

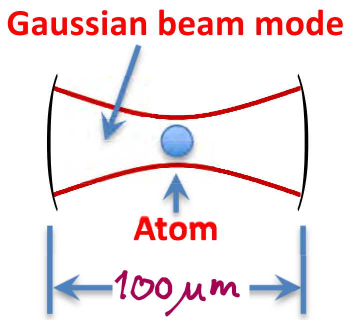
Classical Rabi frequency

mean field $\langle\alpha(t)|\hat{\vec{E}}|\alpha(t)\rangle$

Quantized Light – Matter Interactions

Quantized Light – Matter Interactions

More Cavity QED – Dressed States



$c/2L \gg A_{21}$

$|g_{\vec{k}}| \gg A_{21}, \gamma$

Energy levels of the atom-cavity system

Bare & Dressed States

Return to single - mode result

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} =$$

$$\hbar\omega\hat{a}^{+}\hat{a} + \frac{1}{2}\omega_{21}\hat{\sigma}_z \qquad = H_0$$

$$+\hbar g\left(\hat{\sigma}_{+}\hat{a}e^{i\Delta t} + \hat{\sigma}_{-}\hat{a}^{+}e^{-i\Delta t}\right) \qquad = H_{AF}$$

"Bare" states ($g=0$, eigenstates of H_0)

State	Energy
$\|g,n\rangle$	$E_{g,n} = -\frac{\hbar\omega_{21}}{2} + n\hbar\omega$
$\|e,n-1\rangle$	$E_{e,n} = \frac{\hbar\omega_{21}}{2} + (n-1)\hbar\omega$

Quantized Light – Matter Interactions

"Bare" states ($g=0$, eigenstates of H_0)

State	Energy
$\lvert g,n\rangle$	$E_{g,n} = -\frac{\hbar\omega_{21}}{2} + n\hbar\omega$
$\lvert e,n-1\rangle$	$E_{e,n} = \frac{\hbar\omega_{21}}{2} + (n-1)\hbar\omega$

$\lvert 2,n-1\rangle$

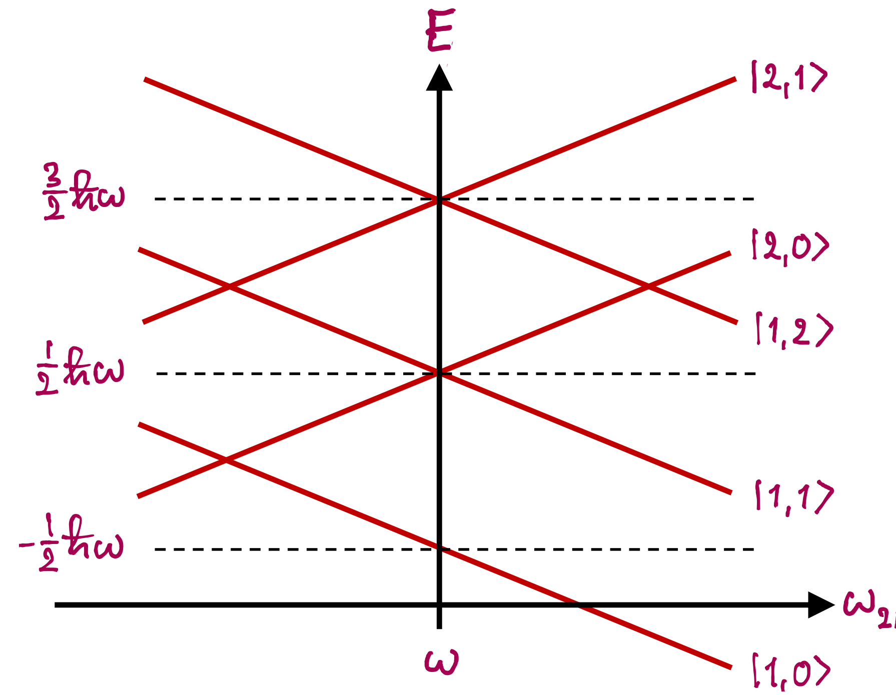
$\lvert e,0\rangle$ $\lvert g,0\rangle$ $\lvert g,1\rangle$ $\lvert g,2\rangle$

$\lvert e,0\rangle$

$\lvert e,n\rangle$

Imagine we can tune ω_{21}

Energy level diagram



Crossings @ $\omega = \omega_{21}$ are degeneracies of pairs with n shared excitations

$n=0$	$\lvert g,0\rangle$
$n=1$	$\{\lvert g,1\rangle, \lvert e,0\rangle\}$
$n=2$	$\{\lvert g,2\rangle, \lvert e,1\rangle\}$
$\vdots$	$\vdots$
n	$\{\lvert g,n\rangle, \lvert e,n-1\rangle\}$

Quantized Light – Matter Interactions

"Bare" states ($g=0$, eigenstates of H_0)

State	Energy
$\|g,n\rangle$	$E_{g,n} = -\dfrac{\hbar\omega_{21}}{2} + n\hbar\omega$
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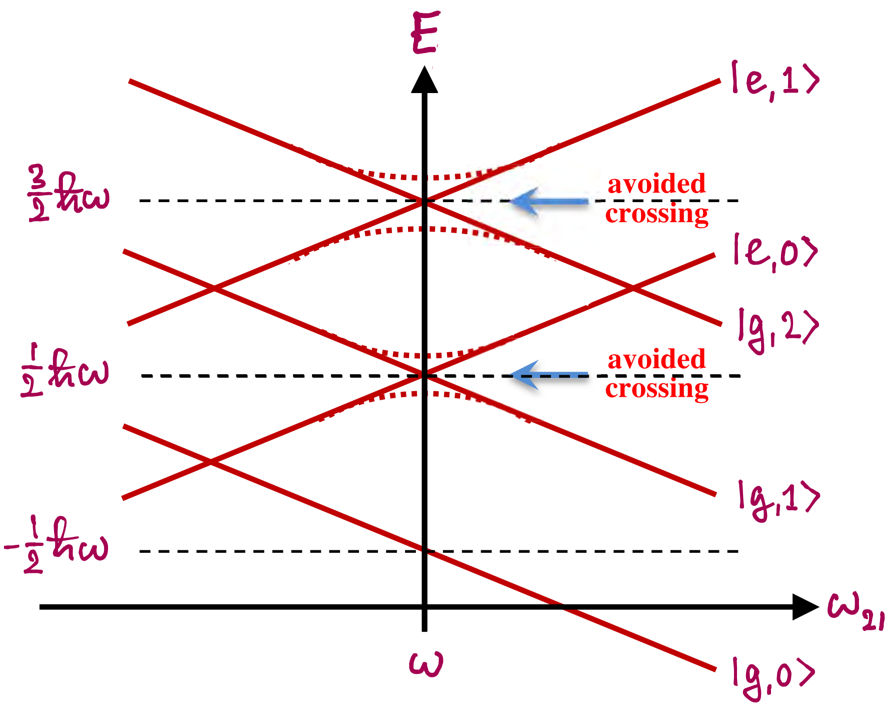
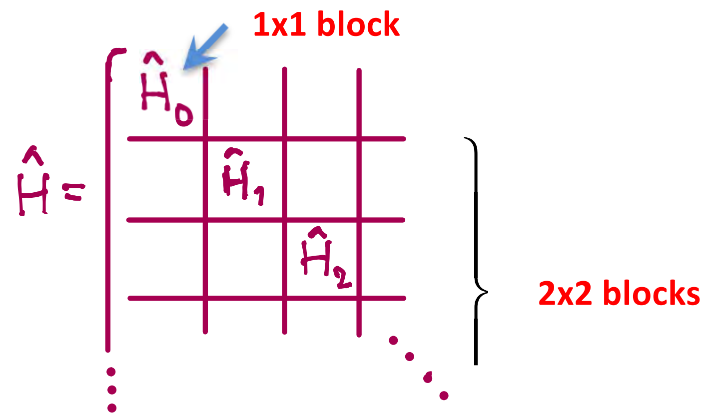
Energy level diagram



"Dressed" states $\left\{ \begin{array}{l} \text{eigenstates of} \\ \hat{H} = \hat{H}_0 + \hat{H}_{AF} \end{array} \right.$

Structure of $\hat{H}$:



Can write this on the form

$$\boxed{\hat{H}_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (n-\tfrac{1}{2})\hbar\omega + \begin{bmatrix} -\hbar\Delta/2 & \hbar g\sqrt{n} \\ \hbar g\sqrt{n} & \hbar\Delta/2 \end{bmatrix}}$$

$\Delta = \omega_{21} - \omega$

Quantized Light – Matter Interactions

"Dressed" states $\left\{ \begin{array}{l} \text{eigenstates of} \\ \hat{H} = \hat{H}_0 + \hat{H}_{AF} \end{array} \right.$

Structure of $\hat{H}$ **:**

1x1 block

$$\hat{H} = \begin{bmatrix} \hat{H}_0 & & & \\ & \hat{H}_1 & & \\ & & \hat{H}_2 & \\ \vdots & & & \ddots \end{bmatrix}$$

2x2 blocks

Can write this on the form

$$\boxed{\hat{H}_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (n - \tfrac{1}{2})\hbar\omega + \begin{bmatrix} -\hbar\Delta/2 & \hbar g\sqrt{n} \\ \hbar g\sqrt{n} & \hbar\Delta/2 \end{bmatrix}}$$

$\Delta = \omega_{21} - \omega$

Eigenvalues $E_{\pm} = (n - \tfrac{1}{2})\hbar\omega \pm \frac{\hbar}{2}\sqrt{4g^2 n + \Delta^2}$

Eigenstates

$$|+,n\rangle = \begin{matrix}\cos \\ \sin\end{matrix}\left(\theta_n/2\right)|1,n\rangle + \begin{matrix}\sin \\ \cos\end{matrix}\left(\theta_n/2\right)|2,n-1\rangle$$

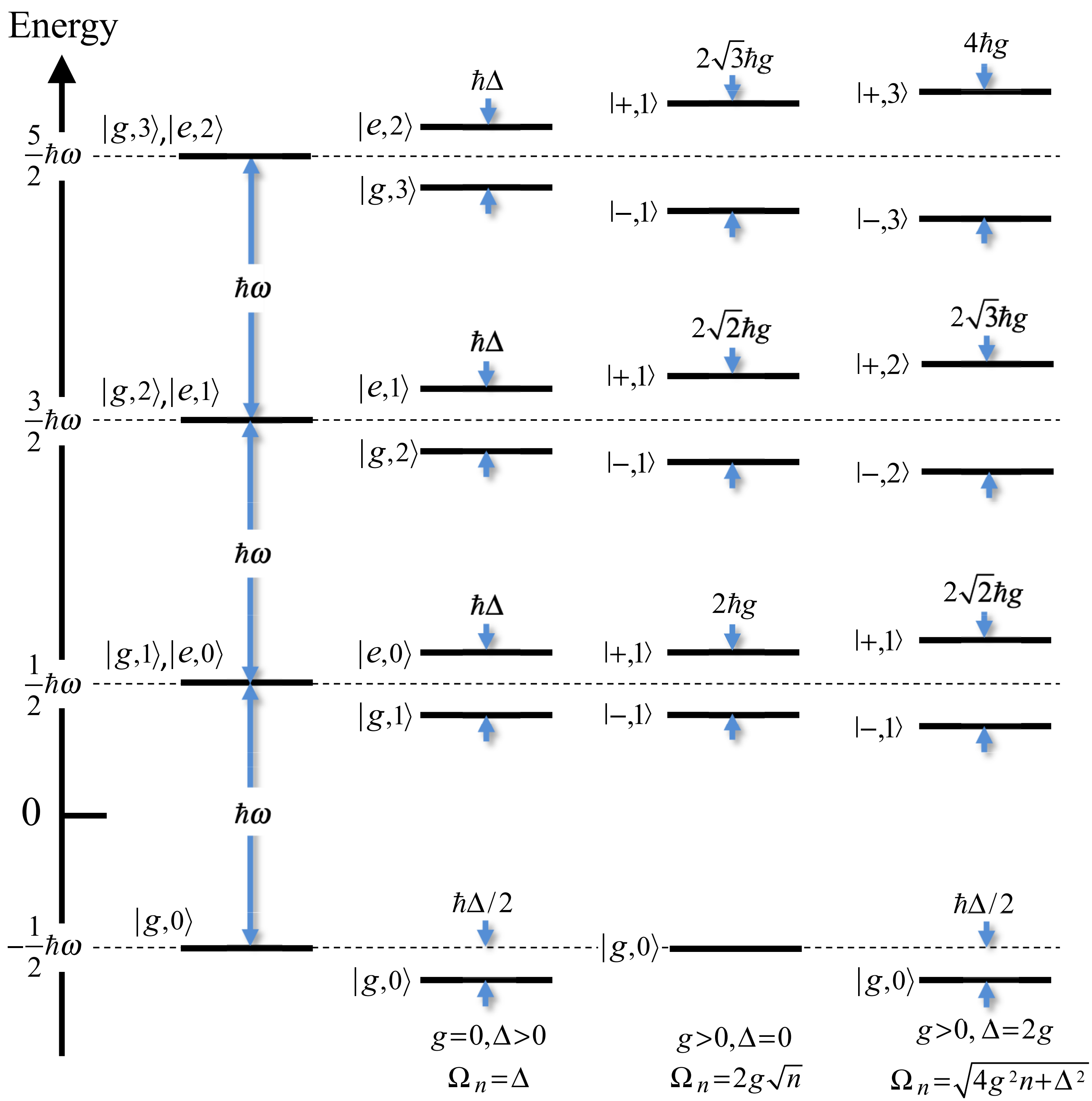
$$|-,n\rangle = -\begin{matrix}\sin \\ \cos\end{matrix}\left(\theta_n/2\right)|1,n\rangle + \begin{matrix}\cos \\ \sin\end{matrix}\left(\theta_n/2\right)|2,n-1\rangle$$

for $\begin{matrix} \Delta \leq 0 \\ \Delta > 0 \end{matrix}$

Mixing angle $\tan\theta_n = -\frac{2g\sqrt{n}}{\Delta}$

Energy Spectrum?

The Jaynes-Cummings Ladder $E_{\pm} = (n - \frac{1}{2})\hbar\omega \pm \hbar\Omega_n$

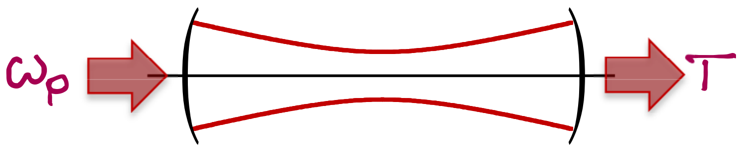


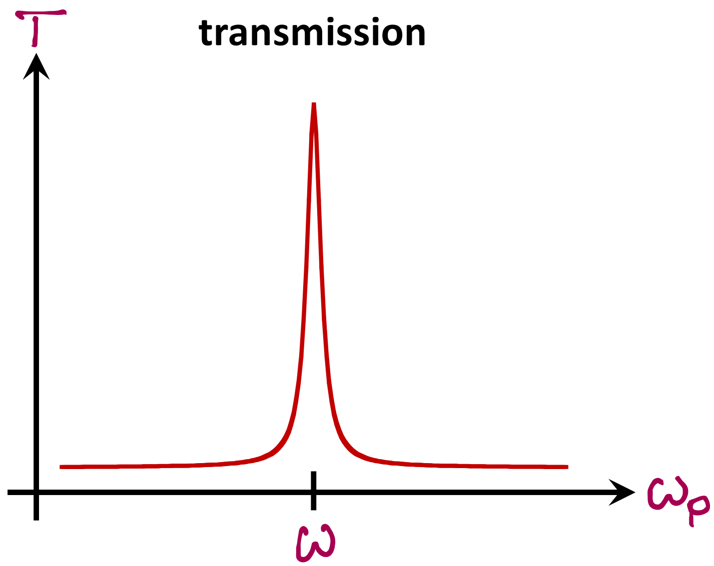
Phenomena Rooted in the Jaynes-Cummings Ladder

Vacuum Rabi Splitting

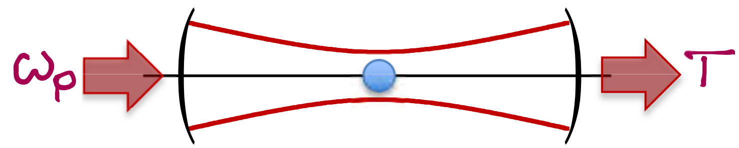
Consider the following experiments

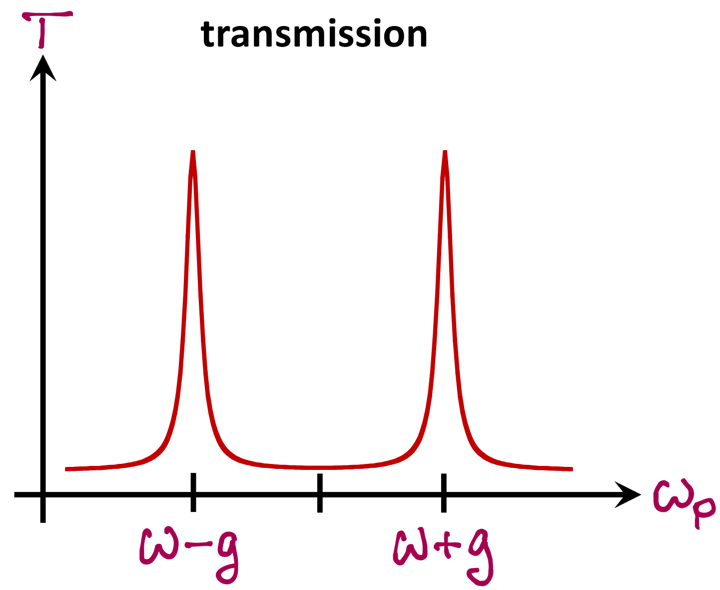
empty Cavity

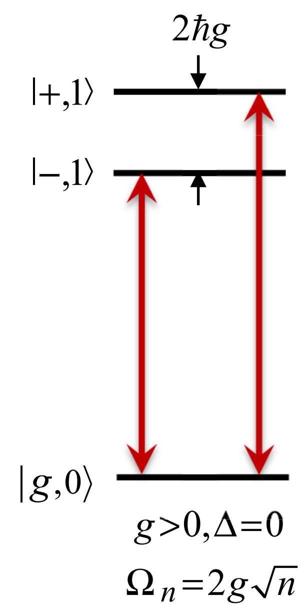




Cavity + Atom, Δ = 0,
one collective exciation



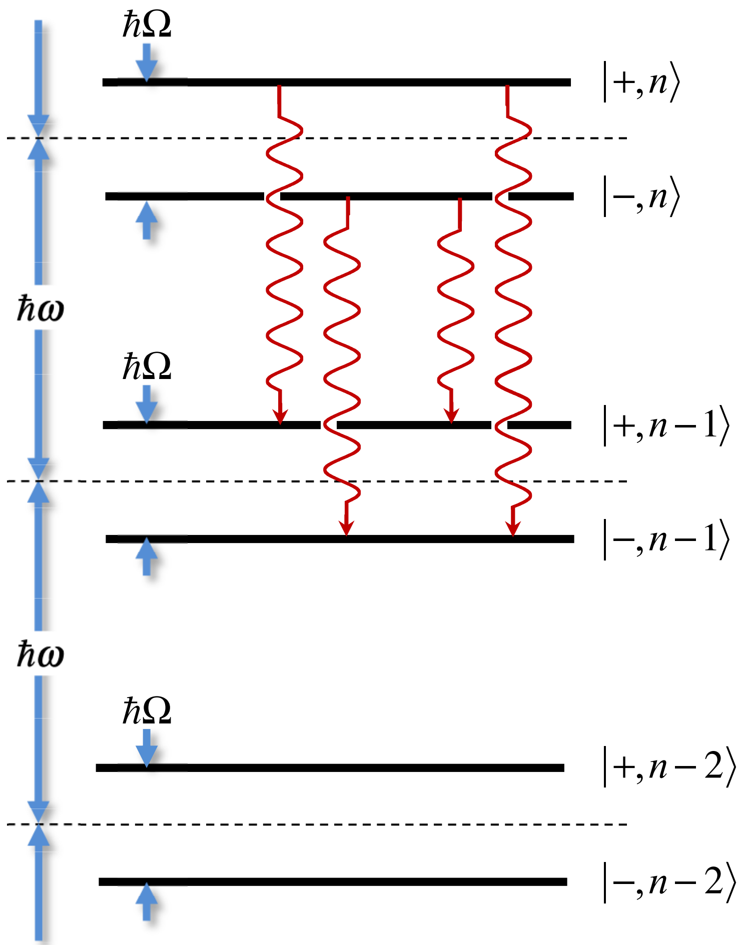




Fluorescence - Mollow Triplet

Fluorescence - Mollow Triplet

For the coherent state $|\alpha\rangle$ **let the mode volume** V **and mean photon number** $\bar{n}$ **go to infinity s.t.** $V \times \bar{n} \sim$ **constant. Then** $g \sim$ **constant, and thus** $\Omega^2 = 4g^2(\bar{n}+\sqrt{\bar{n}}) + \Delta^2 \sim 4g^2\bar{n} + \Delta^2$ **.**



Fluorescence - Mollow Triplet

For the coherent state $|\alpha\rangle$ let the mode volume V and mean photon number $\bar{n}$ go to infinity s.t. $V \times \bar{n} \sim$ constant. Then $g \sim$ constant, and thus $\Omega^2 = 4g^2(\bar{n} + \sqrt{\bar{n}}) + \Delta^2 \sim 4g^2\bar{n} + \Delta^2$.