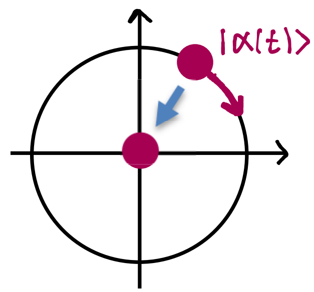
Quantum States of the Quantized Field

Odds and Ends – Quantum-Classical Correspondence

Define a Translation Operator

$$\hat{T}_\alpha(t) = e^{\alpha^* e^{i\omega t}\hat{a} - \alpha e^{-i\omega t}\hat{a}^+} = \hat{D}(-\alpha e^{-i\omega t})$$

This translates a coherent state to the vacuum



We can use the relation

$$[\hat{a}, \hat{F}(\hat{a}^+)] = dF(\hat{a}^+)/d\hat{a}^+$$

to show that

$$[\hat{a}, \hat{T}_\alpha] = \hat{a}\hat{T}_\alpha - \hat{T}_\alpha\hat{a} = -\alpha e^{-i\omega t} T_\alpha$$

We can rearrange and show that

$$\hat{T}_\alpha\hat{a} = \hat{a}\hat{T}_\alpha + \alpha e^{-i\omega t}\hat{T}_\alpha$$

$$\Rightarrow \hat{T}_\alpha\hat{a}\hat{T}_\alpha^+ = \hat{a} + \alpha e^{-i\omega t}$$

Field Observable ($\hat{a}$)

C-valued dimensionless field amplitude ($\alpha e^{-i\omega t}$)

From this we get

$$\underbrace{\hat{T}_\alpha \hat{E}_\perp \hat{T}_\alpha^+}_{\hat{E}'_\perp} = \hat{T}_\alpha\left(\mathcal{E}_k \hat{a} e^{i\vec{k}\cdot\vec{r}} + H.C.\right)\hat{T}_\alpha^+$$

$$+ \mathcal{E}_k \alpha e^{-i(\omega t - \vec{k}\cdot\vec{r})} + C.C.$$

New Field Observable ($\hat{E}'_\perp$)

$E_\perp^{Cl}(\alpha, t)$ Classical Field

Overall, the unitary transformation $\hat{T}_\alpha(t)$ implements the map

$$\boxed{\begin{gathered}\hat{T}_\alpha(t)\hat{E}_\perp\hat{T}_\alpha(t)^+ = \hat{E}_\perp + E_\perp^{Cl}(\alpha, t) \\ |\psi'(t)\rangle = T_\alpha(t)|\alpha(t)\rangle = |0\rangle\end{gathered}}$$

We can work with

$\hat{E}_\perp, |\alpha(t)\rangle$ **or** $\hat{E}_\perp + E_\perp^{Cl}(\alpha, t),\ |0\rangle$

Validates Semiclassical Optics for strong Coherent Fields!

Quantized Light – Matter Interactions

General Problem:

Quantum Atom ⟷ Quantized Modes Of EM Field

Electric Dipole Interaction

Starting Point: System Hamiltonian

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} \qquad (1)$$

$$\hat{H}_F = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \left(\hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} + 1/2\right) \qquad \text{Field}$$

$$\hat{H}_A = \sum_i E_i |i\rangle\langle i| = \sum_i E_i \hat{\sigma}_{ii} \qquad \text{Atom}$$

$$\hat{H}_{AF} = -\hat{\vec{\mu}} \cdot \hat{\vec{E}}(\vec{r},t) \qquad \text{ED interaction}$$

E_i, $|i\rangle$: energies, energy levels of the atom

Dipole Operator:

$$\hat{\vec{\mu}} = \sum_{i,j} \vec{\mu}_{ij} |i\rangle\langle j| = \sum_{i,j} \vec{\mu}_{ij} \hat{\sigma}_{ij} \qquad (2)$$

Field Operator:

$$\hat{\vec{E}}(\vec{r},t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + H.c., \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar\omega_{\vec{k}}}{2\varepsilon_0 V}}$$

2 polarization modes implicit

Pin down atom where $u_{\vec{k}}(\vec{r}) = 1$

– anywhere if $u_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$

– if $u_{\vec{k}}(\vec{r}) = \sin(kz)$ then where $\sin(kz) = 1$

$$\hat{\vec{E}}(\vec{r},t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \left(\hat{a}_{\vec{k}} + \hat{a}^+_{\vec{k}}\right) \qquad (3)$$

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Dipole Operator:

(2) $$\boxed{\hat{\vec{\mu}} = \sum_{i,j} \vec{\mu}_{ij} |i\rangle\langle j| = \sum_{i,j} \vec{\mu}_{ij} \hat{\sigma}_{ij}}$$

Field Operator:

$$\vec{E}(\vec{r},t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + \text{H.C.}, \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar\omega_{\vec{k}}}{2\varepsilon_0 V}}$$

2 polarization modes implicit

Pin down atom where $u_{\vec{k}}(\vec{r}) = 1$

– **anywhere if** $u_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$

– **if** $u_{\vec{k}}(\vec{r}) = \sin(kz)$ **then where** $\sin(kz) = 1$

(3) $$\boxed{\hat{\vec{E}}(\vec{r},t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})}$$

Combining (2) & (3):

$$\hat{H}_{AF} = \sum_{ij} \sum_{\vec{k}} -\vec{\mu}_{ij} \cdot \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})$$

$$= \sum_{ij} \sum_{\vec{k}} \hbar g^{(ij)}_{\vec{k}} \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})$$

where $\hbar g^{(ij)}_{\vec{k}} = \vec{\mu}_{ij} \cdot \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}}$

Rabi Freq., same sign convention as when we first looked at the Rabi problem

2-level atom ⇨ $(i,j) = (1,2)$ **:**

$$\boxed{\hat{H}_{AF} = \hbar \sum_{\vec{k}} (g_{\vec{k}} \hat{\sigma}_{21} + g^{*}_{\vec{k}} \hat{\sigma}_{12}) (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})}$$

Define:

$$\left.\begin{aligned} \hat{\sigma}_{+} &= \hat{\sigma}_{21} = |2\rangle\langle 1| \\ \hat{\sigma}_{-} &= \hat{\sigma}_{12} = |1\rangle\langle 2| \\ \hat{\sigma}_{z} &= \hat{\sigma}_{22} - \hat{\sigma}_{11} = |2\rangle\langle 2| - |1\rangle\langle 1| \end{aligned}\right\} \Rightarrow$$

Pauli matrices

$$\hat{\sigma}_{x} = \frac{1}{2}(\hat{\sigma}_{+} + \hat{\sigma}_{-})$$

$$\hat{\sigma}_{y} = \frac{1}{2i}(\hat{\sigma}_{+} - \hat{\sigma}_{-})$$

$$\hat{\sigma}_{z}$$

Quantized Light – Matter Interactions

Combining (2) & (3):

$$\hat{H}_{AF} = \sum_{ij}\sum_{\vec{k}} -\vec{\mu}_{ij}\cdot\vec{\varepsilon}_{\vec{k}}\,\mathcal{E}_{\vec{k}}\,\hat{\sigma}_{ij}\,(\hat{a}_{\vec{k}}+\hat{a}^{+}_{\vec{k}})$$

$$= \sum_{ij}\sum_{\vec{k}} \hbar g^{(ij)}_{\vec{k}}\,\hat{\sigma}_{ij}\,(\hat{a}_{\vec{k}}+\hat{a}^{+}_{\vec{k}})$$

where $\hbar g^{(ij)}_{\vec{k}} = \vec{\mu}_{ij}\cdot\vec{\varepsilon}_{\vec{k}}\,\mathcal{E}_{\vec{k}}$

Rabi Freq., same sign convention as when we first looked at the Rabi problem

2-level atom ⇨ $(i,j)=(1,2)$:

$$\boxed{\hat{H}_{AF} = \hbar\sum_{\vec{k}}\left(g_{\vec{k}}\hat{\sigma}_{21}+g^{*}_{\vec{k}}\hat{\sigma}_{12}\right)(\hat{a}_{\vec{k}}+\hat{a}^{+}_{\vec{k}})}$$

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$$\hat{\sigma}_{z}$$

With this notation **(4)**

$$\hat{H}_{AF} = \hbar\sum_{\vec{k}}\left(g_{\vec{k}}\hat{\sigma}_{+}\hat{a}_{\vec{k}} + \cancel{g_{\vec{k}}\hat{\sigma}_{+}\hat{a}^{+}_{\vec{k}}} + \cancel{g_{\vec{k}}\hat{\sigma}_{-}\hat{a}_{\vec{k}}} + g_{\vec{k}}\hat{\sigma}_{-}\hat{a}^{+}_{\vec{k}}\right)$$

Energy conservation?

$$\hat{H}_{AF} = \hbar\sum_{\vec{k}}\left(g_{\vec{k}}\hat{\sigma}_{+}\hat{a}_{\vec{k}} + g_{\vec{k}}\hat{\sigma}_{-}\hat{a}^{+}_{\vec{k}}\right)$$

Putting it all together

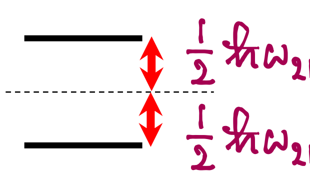
(5)

$$\boxed{\begin{aligned}\hat{H} &= \hat{H}_{F}+\hat{H}_{A}+\hat{H}_{AF} = \\ &\sum_{\vec{k}}\hbar\omega_{\vec{k}}\hat{a}^{+}_{\vec{k}}\hat{a}_{\vec{k}} + \tfrac{1}{2}\hbar\omega_{21}\hat{\sigma}_{z} + \sum_{\vec{k}}\hbar\left(g_{\vec{k}}\hat{\sigma}_{+}\hat{a}_{\vec{k}} + g_{\vec{k}}\hat{\sigma}_{-}\hat{a}^{+}_{\vec{k}}\right)\end{aligned}}$$

We changed the zero point for energy by subtracting

$\sum_{\vec{k}}\frac{1}{2}\hbar\omega_{\vec{k}}$ and $\frac{1}{2}(E_2-E_1)$

field atom



Quantized Light – Matter Interactions

With this notation **(4)**

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Putting it all together

$$\boxed{\begin{aligned} &\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} = \qquad \textbf{(5)} \\ &\sum_{\vec{k}} \hbar \omega_{\vec{k}} \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} + \frac{1}{2} \hbar \omega_{21} \hat{\sigma}_z + \sum_{\vec{k}} \hbar \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right) \end{aligned}}$$

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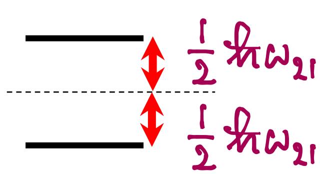
field **atom**



Quantized Light – Matter Interactions

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Putting it all together

(5)

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} =$$

$$\sum_{\vec{k}} \hbar \omega_{\vec{k}} \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} + \frac{1}{2} \hbar \omega_{21} \hat{\sigma}_z + \sum_{\vec{k}} \hbar \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right)$$

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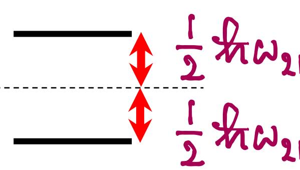

Foundational result for remainder of course

Quantized Light – Matter Interactions

With this notation **(4)**

$$\hat{H}_{AF} = \hbar\sum_{\vec{k}}\left(g_{\vec{k}}\hat{\sigma}_+\hat{a}_{\vec{k}} + \cancel{g_{\vec{k}}\hat{\sigma}_+\hat{a}^\dagger_{\vec{k}}} + \cancel{g_{\vec{k}}\hat{\sigma}_-\hat{a}_{\vec{k}}} + g_{\vec{k}}\hat{\sigma}_-\hat{a}^\dagger_{\vec{k}}\right)$$

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Putting it all together

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$$\sum_{\vec{k}}\hbar\omega_{\vec{k}}\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_z + \sum_{\vec{k}}\hbar\left(g_{\vec{k}}\hat{\sigma}_+\hat{a}_{\vec{k}} + g_{\vec{k}}\hat{\sigma}_-\hat{a}^\dagger_{\vec{k}}\right)$$

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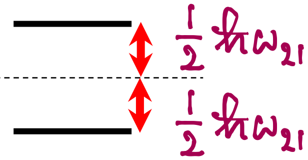
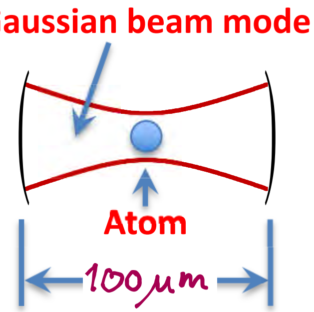


Interaction with Single-mode Fields

Good approx. in small, high-Q Cavity



$c/2L \gg A_{21}$

$|g_{\vec{k}}| \gg A_{21}, \gamma$

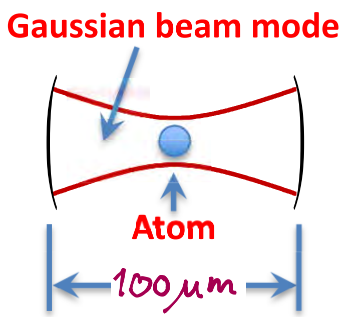
Single-mode (Jaynes-Cummings) Hamiltonian

$$\hat{H} = \underbrace{\hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar g(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger)}_{H_{AF}}$$

Quantized Light – Matter Interactions

Interaction with Single-mode Fields

Good approx. in small, high-Q Cavity



$c/2L \gg A_{21}$

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Single-mode (Jaynes-Cummings) Hamiltonian

$$\boxed{\hat{H} = \underbrace{\hbar\omega\,\hat{a}^{+}\hat{a} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_{z}}_{H_0} + \underbrace{\hbar g(\hat{\sigma}_{+} + \hat{\sigma}_{-})(\hat{a} + \hat{a}^{+})}_{H_{AF}}}$$

Paradigm for spin-1/2 coupled to QHO

- Atom in high-Q cavity *)
- Quantum dot in high-Q
- Rydberg atom in superconducting μw Cavity
- Superconducting qubit in superconducting μw cavity
- Superconducting qubit in superconducting μw stripline cavity (circuit QED)
- Trapped ion with quantized COM motion *)

*) Nobel Prize in Physics 2012