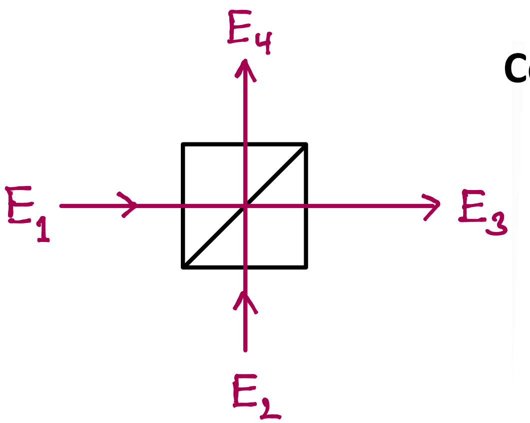
Application: Classical & Quantum Beamsplitters

Classical Beamsplitter



Coupled H & V modes

Linear symmetric input-output map

$$E_3 = tE_1 + rE_2$$
$$E_4 = rE_1 + tE_2$$

Energy conservation requires

$$|E_1|^2 + |E_2|^2 = |E_3|^2 + |E_4|^2$$

Choose $E_1 = E_0 \quad E_2 = 0$

$$|E_3|^3 + |E_4|^2 = E_0\left(|t|^2 + |r|^2\right)$$

Choose $E_1 = \frac{1}{\sqrt{2}} E_0 \quad E_2 = \frac{1}{\sqrt{2}} E_0$

$$|E_3|^2 + |E_4|^2 = \frac{1}{2} E_0 |t+r|^2$$

$$|t|^2 + |r|^2 + tr^* + rt^* = 1$$

From this it follows that

$$\boxed{\begin{aligned} |t|^2 + |r|^2 &= 1 \\ tr^* + rt^* &= 0 \end{aligned}}$$

Classical input-output map

$$\boxed{\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}}$$

Quantum Beamsplitter

Heisenberg Picture ➡ **Field Operators obey Maxwells Eqs**

Classical field	Quantum equivalent
$E_\perp(\vec{r}_\perp, t) \propto \alpha(t)$	$\hat{E}_\perp^{(+)}(\vec{r}_\perp, t) \propto \hat{a}(t)$

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From this it follows that

$$\boxed{\begin{aligned} |t|^2+|r|^2 &= 1 \\ tr^*+rt^* &= 0 \end{aligned}}$$

Classical input-output map

$$\boxed{\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}}$$

Quantum Beamsplitter

Heisenberg Picture ⇒ Field Operators obey Maxwells Eqs

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$$E_\perp(\vec{r},t) \propto \alpha(t)$$

Quantum equivalent

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Quantum Beamsplitter

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⇓

Quantum input-output map

$$\boxed{\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}}$$

Invert Map

$$\begin{aligned} \hat{a}_3 &= t\hat{a}_1 + r\hat{a}_2 \\ \hat{a}_4 &= r\hat{a}_1 + t\hat{a}_2 \end{aligned} \Rightarrow \begin{aligned} \hat{a}_1 &= t^*\hat{a}_3 + r^*\hat{a}_4 \\ \hat{a}_2 &= r^*\hat{a}_3 + t^*\hat{a}_4 \end{aligned}$$

Switch to creation operators ⇒

$$\boxed{\begin{aligned} \hat{a}_1^+ &= t\hat{a}_3^+ + r\hat{a}_4^+ \\ \hat{a}_2^+ &= r\hat{a}_3^+ + t\hat{a}_4^+ \end{aligned}}$$

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Quantum Beamsplitter

$$\begin{pmatrix} \hat{E}_3 \\ \hat{E}_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \hat{E}_1 \\ \hat{E}_2 \end{pmatrix}$$

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Invert Map

$$\hat{a}_3 = t\hat{a}_1 + r\hat{a}_2 \qquad \hat{a}_1 = t^*\hat{a}_3 + r^*\hat{a}_4$$

$$\hat{a}_4 = r\hat{a}_1 + t\hat{a}_2 \qquad \hat{a}_2 = r^*\hat{a}_3 + t^*\hat{a}_4$$

Switch to creation operators

$$\boxed{\begin{aligned} \hat{a}_1^\dagger &= t\hat{a}_3^\dagger + r\hat{a}_4^\dagger \\ \hat{a}_2^\dagger &= r\hat{a}_3^\dagger + t\hat{a}_4^\dagger \end{aligned}}$$

Switch to Schrödinger Picture

General input state: 2-mode vacuum

$$\boxed{|\psi_{in}\rangle = \sum_{nm} f_n \frac{1}{\sqrt{n!}} (\hat{a}_1^\dagger)^n g_m \frac{1}{\sqrt{m!}} (\hat{a}_2^\dagger)^m |0\rangle}$$

The BS maps $\hat{a}_1^\dagger, \hat{a}_2^\dagger$ to linear combinations of $\hat{a}_3^\dagger, \hat{a}_4^\dagger$

General output state: (Schrödinger Picture)

$$\boxed{|\psi_{out}\rangle = \sum_{nm} f_n \frac{1}{\sqrt{n!}} (t\hat{a}_3^\dagger + r\hat{a}_4^\dagger)^n g_m \frac{1}{\sqrt{m!}} (r\hat{a}_3 + t\hat{a}_4)^m |0\rangle}$$

Example: One-photon input state

$$|\psi_{in}\rangle = |1\rangle_1 |0\rangle_2 = \hat{a}_1^\dagger |0\rangle$$

$$|\psi_{out}\rangle = (t\hat{a}_3^\dagger + r\hat{a}_4^\dagger)|0\rangle = t|1\rangle_3|0\rangle_4 + r|0\rangle_3|1\rangle_4$$

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Switch to Schrödinger Picture

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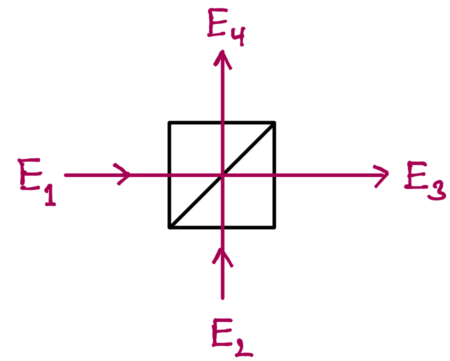
50/50 Beamsplitter

$t = 1/\sqrt{2}, \; r = i/\sqrt{2}$



$$\boxed{|\psi_{out}\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle_3|0\rangle_4 + i|0\rangle_3|1\rangle_4\right)}$$

Note: This is a Photon number-Mode Entangled State

(*) A coherent superposition of states w/ one photon in port 3 and zero in port 4, and zero in port 3 and one in port 4.

Can we assign states such as, e. g.

$\cancel{\frac{1}{\sqrt{2}}(|1\rangle_3 + i|0\rangle_3)}$ to port 3

$\cancel{\frac{1}{\sqrt{2}}(|0\rangle_4 + i|1\rangle_4)}$ to port 4

!

Viewed on their own, each port is in a mixed state

Application: Classical & Quantum Beamsplitters

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Measurement of Subpicosecond Time Intervals between Two Photons by Interference

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(Received 10 July 1987)

A fourth-order interference technique has been used to measure the time intervals between two photons, and by implication the length of the photon wave packet, produced in the process of parametric down-conversion. The width of the time-interval distribution, which is largely determined by an interference filter, is found to be about 100 fs, with an accuracy that could, in principle, be less than 1 fs.

PACS numbers: 42.50.Bs, 42.65.Re

The usual way to determine the duration of a short pulse of light is to superpose two similar pulses and to measure the overlap with a device having a nonlinear response.[1] The latter might, for example, make use of the process of harmonic generation in a nonlinear medium. Indeed, such a technique was recently used[2] to determine the coherence length of the light generated in the process of parametric down-conversion.[3] The coherence time was found to be of subpicosecond duration, as predicted theoretically.[4] It is, however, in the nature of the technique that it requires very intense light pulses and would be of no use for the measurement of single

phasized that the signal and idler photons have no definite phase, and are therefore mutually incoherent, in the sense that they exhibit no second-order interference when brought together at detector D1 or D2. However, fourth-order interference effects occur, as demonstrated by the coincidence counting rate between D1 and D2.[6–8] The experiment has some similarities to another, recently reported, two-photon interference experiment in which fringes were observed and measured, but without the use of a beam splitter.[6]

Although the sum frequency $\omega_1+\omega_2$ is very well defined in the experiment, the individual down-shifted

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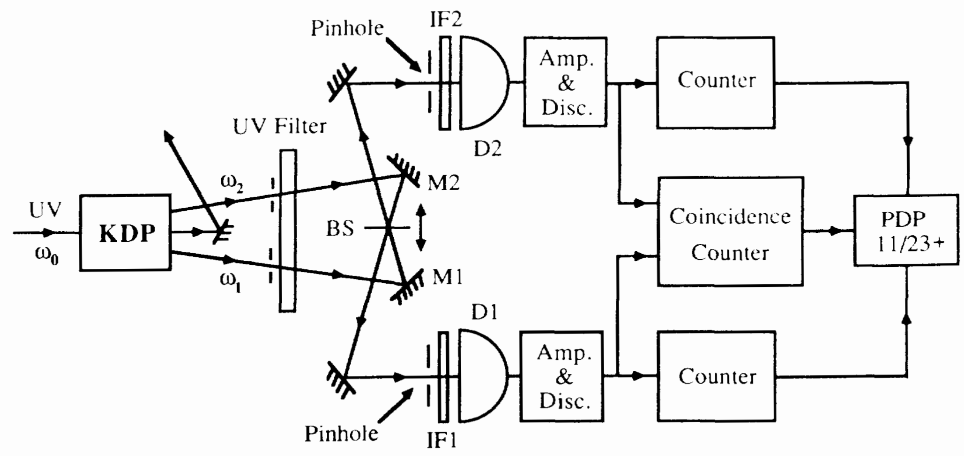
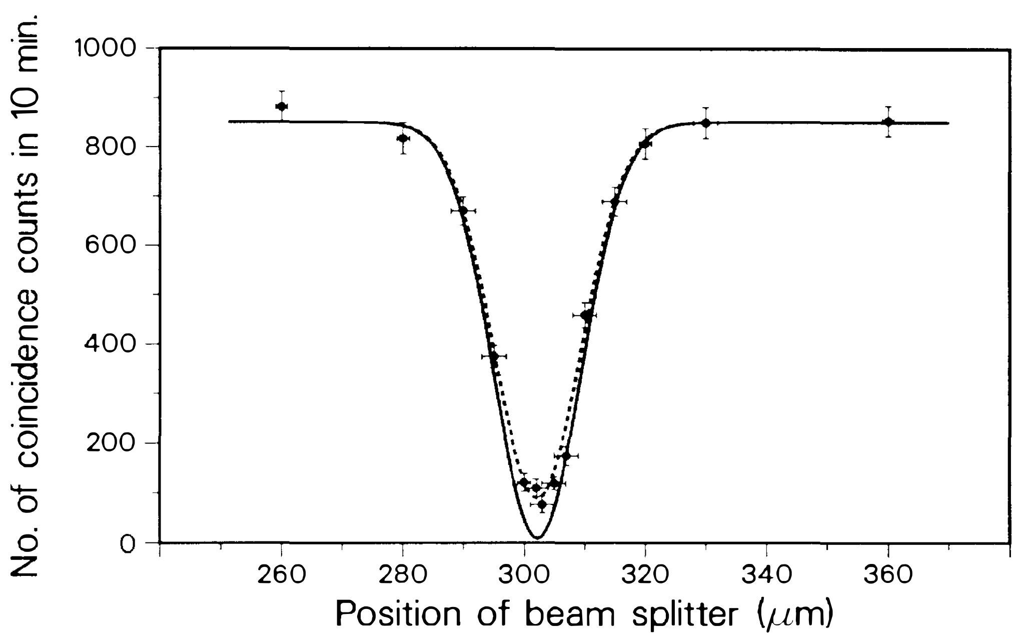


FIG. 1. Outline of the experimental setup.



Quantum Electrodynamics – QED

Quantum States of the Quantized Field

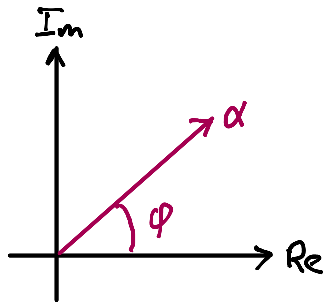
Amplitude and Phase

- Key characteristics of classical fields
- Need equivalents for quantum fields

Classical Field

$$E(z,t) = \mathcal{E}_k \boxed{\alpha}\, e^{-i(\omega t - kz)} + c.c.$$

$|\alpha| e^{i\varphi}$



Quantum Field

$$\hat{E}(z,t) = \mathcal{E}_k \boxed{\hat{a}}\, e^{-i(\omega t - kz)} + H.C.$$

Non-Hermitian!
Separate in amplitude & phase?

Consider operators

$$\hat{a} = (\hat{N}+1)^{1/2}\, \hat{\exp}(i\varphi)$$
$$\hat{a}^{+} = \hat{\exp}(-i\varphi)(\hat{N}+1)^{1/2}$$

"phase" "amplitude"

$$\hat{\exp}(i\varphi) = (\hat{N}+1)^{-1/2}\hat{a}$$
$$\hat{\exp}(-i\varphi) = \hat{a}^{+}(\hat{N}+1)^{-1/2}$$

"Phase operators"

$$\hat{\exp}(i\varphi)\hat{\exp}(-i\varphi) = 1 \qquad \hat{\exp}(i\varphi) = \hat{\exp}(-i\varphi)^{+}$$
$$\hat{\exp}(-i\varphi)\hat{\exp}(i\varphi) = 1 \qquad = \left[\hat{\exp}(-i\varphi)\right]^{-1}$$

- Analogous to classical phases
- Non-Hermitian, NOT observables

Quadrature operators?

$$\hat{\cos}\varphi = \frac{1}{2}\left[\hat{\exp}(i\varphi) + \hat{\exp}(-i\varphi)\right]$$
$$= \frac{1}{2}\left[(\hat{N}+1)^{-1/2}\hat{a} + \hat{a}^{+}(\hat{N}+1)^{-1/2}\right]$$

$$\hat{\sin}\varphi = \frac{1}{2i}\left[\hat{\exp}(i\varphi) - \hat{\exp}(i\varphi)\right]$$
$$= \frac{1}{2i}\left[(\hat{N}+1)^{-1/2}\hat{a} - \hat{a}^{+}(\hat{N}+1)^{-1/2}\right]$$

- Hermitian -> observables
- but ultimately too cumbersome

Let's rewind and try again…

Quantum States of the Quantized Field

"Phase operators"

$$\hat{\exp}(i\varphi)\,\hat{\exp}(-i\varphi) = 1 \qquad \hat{\exp}(i\varphi) = \hat{\exp}(-i\varphi)^{+}$$

$$\hat{\exp}(-i\varphi)\,\hat{\exp}(i\varphi) = 1 \qquad \quad = \left[\hat{\exp}(-i\varphi)\right]^{-1}$$

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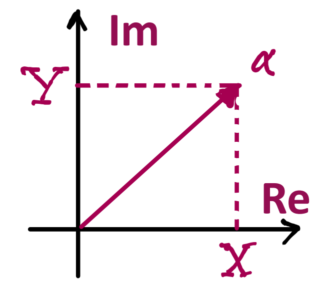
Quadratures of the Classical Field – Take Two

$$E(z,t) = \underbrace{\mathcal{E}_k \alpha_k(t)}\, e^{ikz} + c.c.$$

complex amplitude for mode e^{ikz}



Define

$$\boxed{\begin{aligned} X(t) &= \mathrm{Re}[\alpha_k(t)] = \frac{1}{2}[\alpha_k(t) + \alpha_k^*(t)] = Q(t) \\ Y(t) &= \mathrm{Im}[\alpha_k(t)] = \frac{1}{2i}[\alpha_k(t) - \alpha_k^*(t)] = P(t) \end{aligned}}$$

Quantization: $\alpha \to \hat{a}, \quad \alpha^* \to \hat{a}^+$

$$\left.\begin{aligned} \hat{X}(t) &= \frac{1}{2}\left[\hat{a}_k(t) + \hat{a}_k^{+}(t)\right] = \hat{Q}(t) \\ \hat{Y}(t) &= \frac{1}{2i}\left[\hat{a}_k(t) - \hat{a}_k^{+}(t)\right] = \hat{P}(t) \end{aligned}\right\} \quad [\hat{X}(t), \hat{Y}(t)] = i/2$$

$$\hat{E}(z,t) = \mathcal{E}_k\left(\hat{X}(t) + i\hat{Y}(t)\right)e^{ikz} + H.C.$$
$$= \mathcal{E}_k\left[\hat{X}(t)\cos(kz) - \hat{Y}(t)\sin(kz)\right]$$

– same info, easier to work with –

Quantum States of the Quantized Field

Quadratures of the Classical Field – Take Two

$E(z,t) = \underbrace{\mathcal{E}_k \alpha_k(t)}_{} e^{ikz} + c.c.$

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Quantization: $\alpha \to \hat{a}$, $\alpha^* \to \hat{a}^+$

$$\left.\begin{array}{l}\hat{X}(t) = \frac{1}{2}[\hat{a}_k(t) + \hat{a}_k^+(t)] = \hat{Q}(t) \\ \hat{Y}(t) = \frac{1}{2i}[\hat{a}_k(t) - \hat{a}_k^+(t)] = \hat{P}(t)\end{array}\right\} \quad [\hat{X}(t), \hat{Y}(t)] = i/2$$

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Quantum States of the Field in Mode k

Number States (Foch states)

$$\boxed{\hat{a}^+\hat{a}|n\rangle = n|n\rangle}$$

$$\langle n|\hat{X}|n\rangle = \langle n|\hat{Y}|n\rangle = 0$$
$$\langle n|\hat{X}^2|n\rangle = \langle n|\hat{Y}^2|n\rangle = \frac{1}{2}(n + 1/2)$$

$$\boxed{\Delta X \Delta Y = \frac{1}{2}(n + 1/2)}$$

– HIGHLY non-classical, $\langle \hat{E} \rangle = 0$

– VERY hard to make for large n

Quantum States of the Quantized Field

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Coherent States (Quasi-classical states)

- Closest approximation to classical field
- See Cohen-Tannoudj, complement G_V

Definition: $|\psi\rangle$ is coherent (quasiclassical) iff

$$\langle \hat{X}(t)\rangle = \langle \psi|\hat{X}(t)|\psi\rangle = X(t), \quad \langle \hat{Y}(t)\rangle = Y(t)$$

$$\langle \hat{H}(t)\rangle = \hbar\omega\left(|\alpha(t)|^2 + 1/2\right)$$

noting that

$$\hat{X}(t) \propto \hat{a}(t) = \hat{a}(0)\,e^{-i\omega t}$$

$$\hat{Y}(t) \propto \hat{a}^+(t) = \hat{a}^+(0)\,e^{i\omega t}$$

equivalently

Definition: $|\psi\rangle$ is coherent (quasiclassical) iff

(1) $\langle \hat{a}(0)\rangle = \langle \psi|\hat{a}(0)|\psi\rangle = \alpha(0)$

(2) $\langle \hat{a}^+(0)\,\hat{a}(0)\rangle = \alpha(0)^*\alpha(0)$

Quantum States of the Quantized Field

Coherent States (Quasi-classical states)

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Cohen-Tannoudji, Lecture Notes

equivalently

Definition: a state $|\alpha\rangle$ is coherent iff

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

Finally, one can show

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

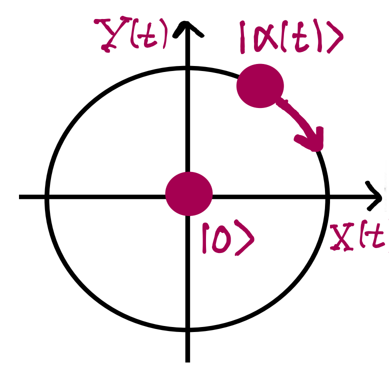
Physical properties

$$\langle\hat{X}(t)\rangle = \mathrm{Re}\left[\alpha(0)e^{-i\omega t}\right]$$

$$\langle\hat{Y}(t)\rangle = \mathrm{Im}\left[\alpha(0)e^{-i\omega t}\right]$$

$$\Delta X(t) = \Delta Y(t) = 1/2$$

$$\Delta X\,\Delta Y = 1/4$$



Quantum States of the Quantized Field

Cohen-Tannoudji, Lecture Notes

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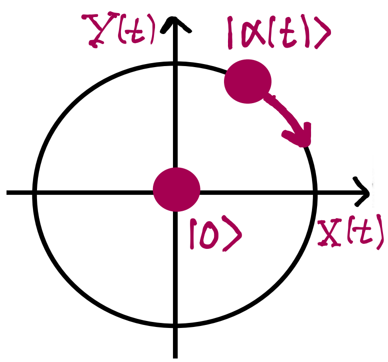

Photon statistics

Measure $\hat{N}$ $\Rightarrow$ $\begin{cases} \text{outcomes } n \\ P(n) = \langle\alpha|n\rangle\langle n|\alpha\rangle = \dfrac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2} \end{cases}$

Poisson distribution w/ $\begin{cases} \text{mean} & \bar{n} = |\alpha|^2 \\ \text{variance} & \Delta n^2 = |\alpha|^2 \end{cases}$

$$\Delta n = \sqrt{\bar{n}}$$ – **Shot Noise**

Quantum States of the Quantized Field

Photon statistics

Measure $\hat{N}$ ⇨ $\begin{cases} \text{outcomes } n \\ P(n) = \langle \alpha | n \rangle\langle n | \alpha \rangle = \dfrac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} \end{cases}$

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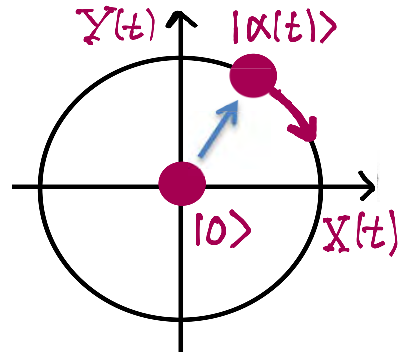
More about Coherent States



Coherent States as translated Vacuum States?

Generating Coherent States from the Vacuum

Definition: $\hat{D}(\alpha) = e^{\alpha \hat{a}^+ - \alpha^* \hat{a}}$

Unitary, equals translation

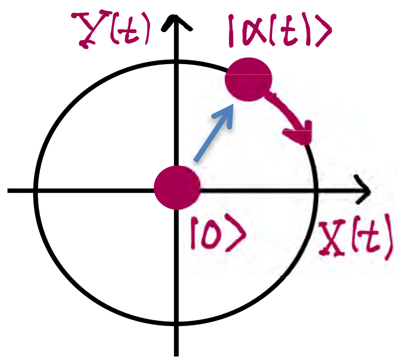
Glaubers formula (from BCH formula)

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{1}{2}[\hat{A},\hat{B}]}$$

for $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$

Quantum States of the Quantized Field

More about Coherent States



$\Rightarrow$ **Coherent States as translated Vacuum States?**

Generating Coherent States from the Vacuum

$$\boxed{\underline{\textbf{Definition}}\textbf{:}\quad \hat{D}(\alpha) = e^{\alpha \hat{a}^{+} - \alpha^{*}\hat{a}}}$$

Unitary, equals translation

Glaubers formula (from BCH formula)

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{1}{2}[\hat{A},\hat{B}]}$$

for $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$

Apply to $[\underbrace{\alpha\hat{a}^{+}}_{\hat{A}}, \underbrace{-\alpha^{*}\hat{a}}_{\hat{B}}] = \underbrace{\alpha^{*}\alpha}_{[\hat{A},\hat{B}]}$

$$\Rightarrow \boxed{\hat{D}(\alpha) = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^{+}} e^{-\alpha^{*}\hat{a}}}$$

Remember: $\hat{a}|0\rangle = 0 \Rightarrow$

$$e^{-\alpha^{*}\hat{a}}|0\rangle = \sum_n \frac{(-\alpha^{*}\hat{a})^n}{n!}|0\rangle = |0\rangle$$

$$\Downarrow$$

$$\hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^{+}}|0\rangle$$

$$= e^{-|\alpha|^2/2} \sum_n \frac{(\alpha\hat{a}^{+})^n}{n!}|0\rangle$$

$$= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle = |\alpha\rangle$$

$$\Downarrow$$

$$\boxed{\hat{D}(\alpha)|0\rangle = |\alpha\rangle}$$

Quantum States of the Quantized Field

Apply to $[\alpha\hat{a}^+, -\alpha^*\hat{a}] = \alpha^*\alpha$

$\hat{A} \uparrow \quad \hat{B} \uparrow \quad [\hat{A},\hat{B}] \uparrow$

$$\Rightarrow \boxed{\hat{D}(\alpha) = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^+} e^{-\alpha^*\hat{a}}}$$

Remember: $\hat{a}|0\rangle = 0 \Rightarrow$

$$e^{-\alpha^*\hat{a}}|0\rangle = \sum_n \frac{(-\alpha^*\hat{a})^n}{n!}|0\rangle = |0\rangle$$

$$\Downarrow$$

$$\hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^+}|0\rangle$$

$$= e^{-|\alpha|^2}\sum_n \frac{(\alpha\hat{a}^+)^n}{n!}|0\rangle$$

$$= e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle = |\alpha\rangle$$

$$\Downarrow$$

$$\boxed{\hat{D}(\alpha)|0\rangle = |\alpha\rangle}$$

OK – $\hat{D}(\alpha)$ generates $|\alpha\rangle$ from the vacuum!

Rewrite:

$$\alpha\hat{a}^+ - \alpha^*\hat{a} = (\alpha-\alpha^*)\hat{X} + i(\alpha+\alpha^*)\hat{Y}$$

$$= i2Y\hat{X} + i2X\hat{Y}$$

where $X = \langle\alpha|\hat{X}|\alpha\rangle, \quad Y = \langle\alpha|\hat{Y}|\alpha\rangle$

Glaubers formula again:

$$\boxed{\hat{D}(\alpha) = e^{i2Y\hat{X}+i2X\hat{Y}} = e^{-XY/4}e^{i2Y\hat{X}}e^{i2X\hat{Y}}}$$

Recall: $\hat{S}(q) = e^{-iq\hat{p}/\hbar} \Rightarrow$ **translation by** q

$\hat{S}(p) = e^{-ip\hat{q}/\hbar} \Rightarrow$ **translation by** p

where $q = q_0 X, \quad p = p_0 Y$

$\hat{q} = q_0\hat{X}, \quad \hat{p} = p_0\hat{Y}$

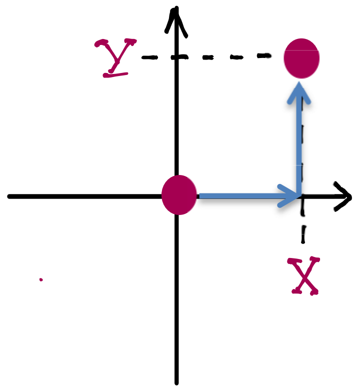
$\& \quad q_0 p_0 = 2\hbar$

Quantum States of the Quantized Field

This gives us

$$\hat{S}(q) = \hat{S}(X) = e^{i2X\hat{Y}}, \quad \hat{S}(p) = \hat{S}(Y) = e^{i2Y\hat{X}}$$

$\hat{D}(\alpha)$ translates along X then Y



Discussion –

How to do this?

Quantum States of the Quantized Field

Coherent States from Classical Dipole Radiation

Classical Dipole $d(t) = d_0 \cos(\omega t)$ @ $t=0$

Quantized Field $\hat{E}(z) = \mathcal{E}_k(\hat{a} + \hat{a}^+)$

Dipole-Field Interaction

$$\hat{H} = \hbar\omega(\hat{a}^+\hat{a} + 1/2) + \hbar\lambda(t)(\hat{a} + \hat{a}^+)$$

$$\lambda(t) = -\frac{d(t)\mathcal{E}_k}{\hbar} = \lambda_0 \cos(\omega t)$$

Drive from $t=0$ to T

$$\alpha(T) = -i\frac{\lambda_0}{2} e^{-i(\omega-\omega')T/2} \frac{\sin[(\omega-\omega')T/2]}{(\omega-\omega')/2}$$

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Recall from Semi-Classical Laser Theory

$\langle \hat{p}(t) \rangle$ **drives** $\hat{E}(t)$

classical dipole + quantum fluctuations

coherent state + quantum fluctuations

For $t > T$ we have a coherent state

$$\alpha(t) = \alpha(T) e^{-i\omega(t-T)}$$

Quantum States of the Quantized Field

Recall from Semi-Classical Laser Theory

$\langle \hat{p}(t) \rangle$ **drives** $\hat{E}(t)$

classical dipole
+ quantum fluctuations

coherent state
+ quantum fluctuations

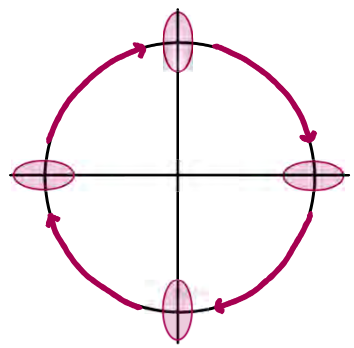
Quantum States of the Quantized Field

Squeezed States

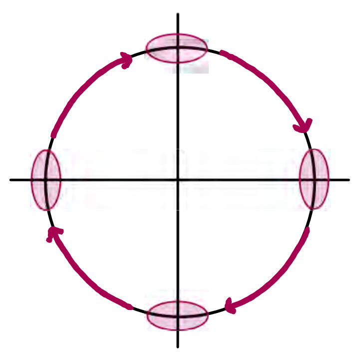
Minimum uncertainty states w/assymmetry

$$\Delta X \, \Delta Y = 1/4 , \quad \Delta X(t) \neq \Delta Y(t)$$

Phase Squeezing



Amplitude Squeezing



Requires interaction with Nonlinear medium

Odds and Ends – Thermal States

$$Z = Tr\left[e^{-\hat{H}/k_B T}\right]$$

$$\boxed{\begin{aligned} \hat{\rho} &= \sum_n P(n) |n\rangle\langle n| = \frac{1}{Z} \sum_n e^{-E_n/k_B T} |n\rangle\langle n| \\ &= (1-q) \sum_n q^n |n\rangle\langle n| , \quad q = e^{-\hbar\omega/k_B T} \end{aligned}}$$

Mean Photon Number:

$$\boxed{\begin{aligned} \bar{n} = Tr(\hat{\rho}\hat{N}) &= \sum_{k,n} \langle k|(1-q) q^n |n\rangle\langle n| \hat{N} |k\rangle \\ &= (1-q) \sum_n n q^n = \frac{q}{1-q} \end{aligned}}$$

Photon Number Uncertainty:

$$\langle \hat{N}^2 \rangle = (1-q) \sum_n n^2 q^n = \frac{q^2+q}{(1-q)}$$

Quantum States of the Quantized Field

Odds and Ends – Thermal States

$$Z = \mathrm{Tr}\left[e^{-\hat{H}/k_B T}\right]$$

$$\hat{\rho} = \sum_n P(n)|n\rangle\langle n| = \frac{1}{Z}\sum_n e^{-E_n/k_B T}|n\rangle\langle n|$$

$$= (1-q)\sum_n q^n |n\rangle\langle n|, \quad q = e^{-\hbar\omega/k_B T}$$

Mean Photon Number:

$$\bar{n} = \mathrm{Tr}(\hat{\rho}\hat{N}) = \sum_{k,n}\langle k|(1-q)q^n|n\rangle\langle n|\hat{N}|k\rangle$$

$$= (1-q)\sum_n n q^n = \frac{q}{1-q}$$

Photon Number Uncertainty:

$$\langle\hat{N}^2\rangle = (1-q)\sum_n n^2 q^n = \frac{q^2+q}{(1-q)}$$

$$\Delta n^2 = \langle\hat{N}^2\rangle - \langle\hat{N}\rangle^2$$

$$= \frac{q^2+q}{(1-q)} - \frac{q^2}{(1-q)^2} = \frac{q}{(1-q)^2}$$

$$\bar{n} = \frac{q}{1-q}$$

Coherent State limit

$$\Delta n = \frac{\sqrt{q}}{1-q} = \sqrt{\bar{n}(\bar{n}+1)} \geq \sqrt{\bar{n}}$$

Optical Frequencies, Room Temperature:

$$\lambda = 1\,\mu m, \quad T = 300\,K$$

$$q = 6.5\times 10^{-6}, \quad \bar{n} \sim 10^{-6}$$

Quantum States of the Quantized Field

Odds and Ends – Quantum-Classical Correspondence

Define a Translation Operator

$$\hat{T}_\alpha(t) = e^{\alpha^* e^{i\omega t}\hat{a} - \alpha e^{-i\omega t}\hat{a}^+} = \hat{D}(-\alpha e^{-i\omega t})$$

Use $[\hat{a}, \hat{F}(\hat{a}^+)] = dF(\hat{a}^+)/d\hat{a}^+$ **to show**

$$[\hat{a}, \hat{T}_\alpha] = \hat{a}\hat{T}_\alpha - \hat{T}_\alpha\hat{a} = -\alpha e^{-i\omega t}\hat{T}_\alpha$$

$\Rightarrow$ $\hat{T}_\alpha\hat{a} = \hat{a}\hat{T}_\alpha + \alpha e^{-i\omega t}\hat{T}_\alpha$

$\Rightarrow$ $\hat{T}_\alpha\hat{a}\hat{T}_\alpha^+ = \hat{a} + \alpha e^{-i\omega t}$

From this we get

(1) Field Observable

$$\hat{E}'_\perp = \underbrace{\hat{T}_\alpha\hat{E}_\perp\hat{T}_\alpha^+} = \hat{T}_\alpha\left(\mathcal{E}_k\hat{a}e^{i\vec{k}\cdot\vec{r}} + H.C.\right)\hat{T}_\alpha^+$$

$$= \mathcal{E}_k\hat{a}e^{i\vec{k}\cdot\vec{r}} + H.C. + \mathcal{E}_k\alpha e^{-i(\omega t - \vec{k}\cdot\vec{r})} + C.C.$$

$$= \hat{E}_\perp + E_\perp^{Cl}(\alpha, t)$$

(2) Classical Field

(3) Field Observable

We also have $|\psi'(t)\rangle = \hat{T}_\alpha|\alpha(t)\rangle = |0\rangle$

Action of the unitary transformation $\hat{T}_\alpha(t)$

$$\hat{E}'_\perp = \hat{T}_\alpha(t)\hat{E}_\perp\hat{T}_\alpha(t)^+ = \hat{E}_\perp + E_\perp^{Cl}(\alpha, t)$$

$$|\psi'(t)\rangle = T_\alpha(t)|\alpha(t)\rangle = |0\rangle$$

We can work with

$\hat{E}_\perp, |\alpha(t)\rangle$ **or** $\hat{E}_\perp + E_\perp^{Cl}(\alpha, t), |0\rangle$

Validates Semiclassical Optics for strong Coherent Fields!

Quantized Light – Matter Interactions

General Problem:

Quantum Atom $\longleftrightarrow$ **Quantized Modes Of EM Field**

Electric Dipole Interaction

Starting Point: System Hamiltonian

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} \qquad \textbf{(1)}$$

$$\hat{H}_F = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \left(\hat{a}^{+}_{\vec{k}} \hat{a}_{\vec{k}} + \tfrac{1}{2} \right) \qquad \textbf{Field}$$

$$\hat{H}_A = \sum_i E_i |i\rangle\langle i| = \sum_i E_i \hat{\sigma}_{ii} \qquad \textbf{Atom}$$

$$\hat{H}_{AF} = -\hat{\vec{p}} \cdot \hat{\vec{E}}(\vec{r},t) \qquad \textbf{ED interaction}$$

E_i, $|i\rangle$: energies, energy levels of the atom