OPTI 544 Final Exam, May 11, 2023 Jessen
Solution Set

Problem I

(a) The Ramsey Method of Separated Oscillatory Fields is a sequence of two short and intense $\pi/2$ pulses that together will transfer a 2-level atom from the south pole to the north pole of the Bloch sphere. If the $\pi/2$ pulses are separated by a long period with no fields present, then any detuning between the atomic transition frequency and the frequency of the driving field will cause the Bloch vector to precess in the equatorial plane between the pulses. If the precession is substantial then the 2nd pulse will not transfer the atom to the excited state, and a subsequent measurement will show reduced excited state population. The "Ramsey trick" is commonly used in atomic clocks, where one wants to compare the frequency of the driving field to the transition frequency of an *unperturbed* atom.

(b) The Hong-Ou-Mandel experiment explores the interference of indistinguishable single photon wave packets. The experiment starts with a spontaneous parametric down conversion event that splits a photon of frequency 2ω into two separate photons at frequency ω. The single-photon wave packets are then recombined on a beam splitter where they interfere, and the statistics of the photons emerging in the beam splitter outputs is measured with avalanche photodetectors. If the input wave packets overlap perfectly on the beams splitter then coincidence detections (one photon emerging in each port) are never observed. This indicates that interference always results in pairs of photons emerging in either one or the other output. This is variously described as "photon bunching" or "photons are bosons". If one wave packet is delayed relative to the other before recombination on the beam splitter, the two photons become distinguishable by their arrival time. This causes the Hong-Ou-Mandel interference to disappear and the photons to appear at random in either output port.

(c) Wheelers Delayed Choice experiment is an experiment that explores the wave-particle duality of photons. At its core is a Mach-Zender interferometer consisting of two beam splitters, the second of which can be inserted or removed. With the 2nd. beam splitter (BS) in place, the photon behaves as a wave and two-path interference is seen when changing the length of one path relative to the other. Without the 2nd BS the photon behaves as a particle, always emerging in one output port or the other with 50/50 probability, irrespective of the path length difference. Central to the experiment is the ability to insert or remove the 2nd BS *at random* and *after* the photon has passed the 1st BS but *before* it arrives at the 2nd BS. This rules out any possibility that the wave or particle nature of the photon is determined by the time it encounters the first BS. That in turn allows us to conclude that a photon is both a particle and wave, and that we always see the property that our measurement is designed to reveal.

Problem II

(a) The Observables associated with energy and photon number are closely related: $\hat{H} = \hbar\omega(\hat{N} + 1/2)$.

The mean energy E of the coherent state is the expectation value of $\hat{H}$, which in turn is $\langle \hat{H} \rangle = \hbar\omega \langle \hat{N} \rangle$, where $\langle \hat{N} \rangle = \overline{n}$ is the mean photon number. We have dropped the zero point energy, in the expectation that it is negligible when the total energy is in the macroscopic domain.

Given the above, we have $\langle \hat{H} \rangle = \hbar\omega \langle \hat{N} \rangle \Rightarrow E = \hbar\omega \times \overline{n} \Rightarrow \overline{n} = E/\hbar\omega$.

First, $$\hbar\omega = 1.05\times 10^{-34} Js \times 2\pi \frac{C}{\lambda} = \frac{2\pi \times 1.05 \times 10^{-34} Js \times 3 \times 10^{8} m s^{-1}}{10^{-6} m} = 1.98 \times 10^{-19} J$$

Second, $$\overline{n} = \frac{E}{\hbar\omega} = \frac{1\times 10^{-12} J}{1.98 \times 10^{-19} J} = 5.05 \times 10^{6}$$

Check: $5.05 \times 10^{6} >> \frac{1}{2}$ so it is OK to drop the zero point energy.

(b) A coherent state is neither an eigenstate of energy nor an eigenstate of photon number. Thus, when given a coherent state, a measurement of $\hat{N}$ will not have a deterministic outcome. Instead, the measurement will return one of many possible values n, of which some are more likely than others.

Frequentist's perspective:
If we prepare a large number of wave packets in identical coherent states $|\alpha\rangle$ and subsequently do a measurement of the photon number on each of them, the outcomes n will will be drawn from a statistical distribution that reflects the admixture of different n-photon states in $|\alpha\rangle$. The mean value of the measurement outcomes will be $\overline{n}$.

Bayesianist's perspective:
If we prepare a wave packet in the coherent state $|\alpha\rangle$ and subsequently do a measurement of the photon number, we know from our prior information (the state is $|\alpha\rangle$) that different outcomes n are possible and will occur with probabilities that reflect the admixture of n-photon states in $|\alpha\rangle$. The expectation (mean) value of the measurement will be $\langle\alpha|\hat{N}|\alpha\rangle = \overline{n}$.

Problem III

(a) Given the coherent state input $|\Psi_{\text{in}}\rangle = |\alpha\rangle_1|0\rangle_2 = e^{\alpha\hat{a}_1^\dagger - \alpha^*\hat{a}_1}|0\rangle$, the output will be

$$|\Psi_{\text{out}}\rangle = e^{t\alpha\hat{a}_3^\dagger - r\alpha^*\hat{a}_4}|0\rangle = |t\alpha\rangle_3|r\alpha\rangle_4$$

The output is a product of a state vector for port 3 and a state vector for port 4, $|\Psi_{\text{out}}\rangle = |t\alpha\rangle_3|r\alpha\rangle_4$, so there is no photon number - mode entanglement.

(b) With an input $|\Psi_{\text{in}}\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}_1^\dagger\right)^n|0\rangle = |n\rangle_1|0\rangle_2$, the general input-output map for the beam splitter tells us that we can write the output as

$$|\Psi_{\text{out}}\rangle = \sum_n \frac{1}{\sqrt{n!}}(t\hat{a}_3^\dagger + r\hat{a}_4^\dagger)^n|0\rangle$$

Using the binomial expansion, we have

$$\begin{aligned}
|\Psi_{\text{out}}\rangle &= \frac{1}{\sqrt{n!}}(t^*\hat{a}_3^\dagger + r^*\hat{a}_4^\dagger)^n|0\rangle = \frac{1}{\sqrt{n!}}\sum_{k=0}^{n}\binom{n}{k}(t^*\hat{a}_3^\dagger)^{n-k}(r^*\hat{a}_4^\dagger)^k|0\rangle \quad (*)\\
&= \frac{1}{\sqrt{n!}}\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}(t^*\hat{a}_3^\dagger)^{n-k}(r^*\hat{a}_4^\dagger)^k|0\rangle\\
&= \sum_{k=0}^{n}\sqrt{\frac{n!}{(n-k)!k!}}(t^*)^{n-k}(r^*)^k\,\frac{(\hat{a}_3^\dagger)^{n-k}}{\sqrt{(n-k)!}}\frac{(\hat{a}_4^\dagger)^k}{\sqrt{k!}}|0\rangle\\
&= \sum_{k=0}^{n}\sqrt{\binom{n}{k}}(t^*)^{n-k}(r^*)^k|n-k\rangle_3|k\rangle_4\ ,
\end{aligned}$$

where we have taken the calculation all the way through. Note, however, that we can tell already from (*) that the output will be a superposition of states with $(n-k)$ photons emerging in port 3 and k photons emerging in port 4.

(c) The output in III(a) is a product state with no correlation between photon numbers in the two ports.

By contrast, III(b) is a very complex superposition state. By inspection, we see that $|\Psi_{\text{out}}\rangle$ contains groupings of states with $n-k$ photons emerging in port 3 and k photons emerging in port 4, in coherent superposition with k photons emerging in port 3 and $n-k$ photons emerging in port 4. Each of these doublet of states have stong photon number-mode entanglement. As a result, there is no way to describe the overall output as a product of a state vector for port 3 and a state vector for port 4, and we conclude that $|\Psi_{\text{out}}\rangle$ is highly photon number-mode entangled.

(d) The result from (a) tells us that a coherent state remains a coherent state when propagating through a lossy medium, here modeled as a beamsplitter.

By contrast, consider what would happen if, in (c), we were to place a photon number resolving detector in port 4, make a photon number measurement, and throw away the outcome. In that case the correlations between photon numbers in the output ports (n photons in port 4 means $n-k$ photons in port 3) will reduce the state emerging in port 3 to an incoherent mixture of photon number states. This tells us that number states do not remain number states when propagating through a lossy medium.