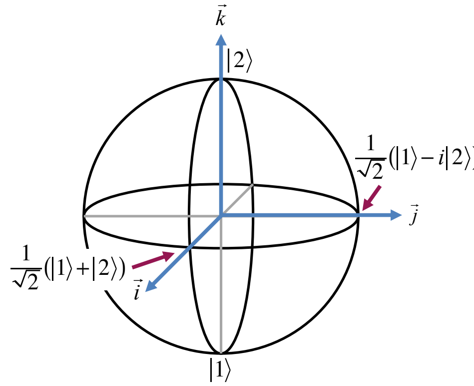
Solution Set, OPTI 544 2nd Midterm Exam, April 12, 2023

Problem I



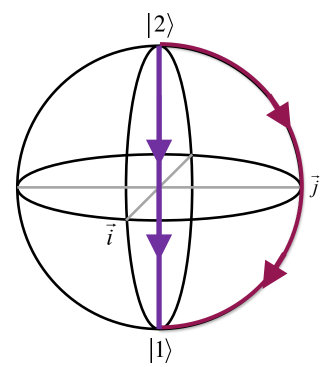
(a) We can copy the Bloch sphere shown on page 3 of the notes "Vector Model..."

(b) If χ is real and positive ($\varphi = 0$) and $\Delta = 0$, the OBE's without relaxation take the form

$$\dot{u} = 0\,, \quad \dot{v} = \chi w\,, \quad \dot{w} = -\chi v$$

(c) With no coherent drive ($\chi = 0$), detuning ($\Delta = 0$), or relaxation other than spontaneous decay ($\beta = A_{21}/2$), the OBE's take the form

$$\dot{u} = -\beta u\,, \quad \dot{v} = -\beta v\,, \quad \dot{w} = -A_{21}(w+1)$$



(d) The OBE's in (b) correspond to a torque $\vec{Q} = -\chi\vec{i}$ that points in the $-\vec{i}$ direction. If we apply a π pulse, a Bloch vector initially pointing to the excited state $|2\rangle$ will rotate clockwise in the $\vec{j}-\vec{k}$ by an angle π and point to the ground state $|1\rangle$. This is the red trajectory on the Bloch sphere at right.

The OBE's in (c) describe a situation where initially $u = v = 0$ and remain there. The last equation tells us that w decays exponentially from $w = 1$ to $w = -1$, moving along the k-axis the entire time. This is the purple trajectory on the Bloch sphere at right.

Problem II

(a) The two-mode input state is $|\Psi_{in}\rangle = \frac{1}{\sqrt{2}}(a_1^+)^2|0\rangle = |2\rangle_1|0\rangle_2$, where $|0\rangle$ is the two-mode vacuum.

(b) The output can be found using the input-output map of the beamsplitter. This allows us to express the creation operators on the input side in terms of creation operators on the output side. Thus,

$$|\Psi_{in}\rangle \to |\Psi_{out}\rangle = \frac{1}{\sqrt{2}}(t\hat{a}_3^+ + r\hat{a}_4^+)^2|0\rangle$$

where we have used $t = 1/\sqrt{2}$, $r = i/\sqrt{2}$. This gives us

$$|\Psi_{out}\rangle = \frac{1}{\sqrt{2}}(t\hat{a}_3^+ + r\hat{a}_4^+)^2|0\rangle = \frac{1}{\sqrt{2}}(t^2\hat{a}_3^+\hat{a}_3^+ + r^2\hat{a}_4^+\hat{a}_4^+ + 2tr\hat{a}_3^+\hat{a}_4^+)|0\rangle = \frac{1}{\sqrt{2}}(\frac{1}{2}\hat{a}_3^+\hat{a}_3^+ - \frac{1}{2}\hat{a}_4^+\hat{a}_4^+ + i\hat{a}_3^+\hat{a}_4^+)|0\rangle$$

$$\Rightarrow \quad |\Psi_{out}\rangle = \frac{1}{2}|2\rangle_3|0\rangle_4 - \frac{1}{2}|0\rangle_3|2\rangle_4 + i\frac{1}{\sqrt{2}}|1\rangle_3|1\rangle_4$$

Sanity check: $|1/2|^2 + |1/2|^2 + |1/\sqrt{2}|^2 = 1 \quad \Rightarrow \quad |\Psi_{out}\rangle$ is normalized.

(c) If we do coincidence counting with photomultiplier detectors as in the Hong-Ou-Mandel experiment, we will find that in 1 out of 4 trials we get a detection in port 3 only, in 1 out of 4 trials we get a detection in port 4 only, and in 2 out of 4 trials we get a coincidence detection. However, we have learned something: We need to send (indistinguishable) photons into both ports if we want the kind of destructive interference that makes the H-O-M experiment interesting. Experiments where photons are sent into one port only tend to be rather boring.

Problem III

(a) From the note set "Introduction to Field Theory" we have

$$\eta(x,t) = \sqrt{L}\sum_k q_k(t)\,u_k(x)$$

where $q_k(t)$ is the generalized coordinate and the normal modes $u_k(x)$ are standing waves in the cavity. The standard Lagrangian has the form

$$L = T - V = \sum_k \frac{1}{2}M\dot{q}_k^2 - \frac{1}{2}M\omega_k^2 q_k^2 = \sum_k L_k$$

where M and ω_k are the linear mass density and normal mode frequency. We can check that the Lagrange equation of motion for any given k reduces to a harmonic oscillator:

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = M\ddot{q}_k^2 - M\omega_k^2 q_k = 0 \Rightarrow \ddot{q}_k^2 - \omega_k^2 q_k = 0$$

(b) The wave equation is

$$\frac{\partial^2 \eta}{\partial t^2} - v^2\frac{\partial \eta^2}{\partial x^2} = 0$$

where $v^2 = \dfrac{Y}{\mu}$

To simplify the notation, we substitute the field for an arbitrary mode k and divide out the factor of $\sqrt{L}$. This gives us

$$\left(\frac{\partial^2}{\partial t^2} - v^2\frac{\partial^2}{\partial x^2}\right) q_k(t)\,u_k(x) = 0 \Rightarrow \ddot{q}_k(t)\,u_k(x) - v^2 q(t)\,u_k''(x) = \ddot{q}(t)_k\,u_k(x) - q_k(t)\,v^2 u_k''(x) = 0$$

Using the hint for $u''(x)$ we then get

$$\ddot{q}_k(t) + \omega_k^2 q_k(t) = 0$$

This is the exact same set of equations as in (a), which gives us confidence that we have found the correct Lagrangian.