Wigner-Weisskopf Theory of Spontaneous Decay

Today: Decay of atomic excited state due to interaction with the quantum electromagnetic field

Setup

Hamiltonian: (Schrödinger Picture)

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} =$$

$$\sum_{\vec{k}} \hbar\omega_{\vec{k}} \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_z + \sum_{\vec{k}} \hbar\left(g_{\vec{k}}\hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}}\hat{\sigma}_- \hat{a}^\dagger_{\vec{k}}\right)$$

Hamiltonian: (Interaction Pict., Res. Approx.)

$$\hat{H}_I(t) = \sum_{\vec{k},\lambda} \hbar g_{\vec{k},\lambda} \hat{\sigma}_+ \hat{a}_{\vec{k},\lambda} e^{i(\omega_{eg}-\omega_{\vec{k}})t} + H.c.$$

S. E.: $i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}_I(t)|\psi_I(t)\rangle$

Expand

$$|\psi(t)\rangle = C_{2,0}(t)|2,0\rangle + \sum_{\vec{k},\lambda} C_{1,1_{\vec{k},\lambda}}(t)|1,1_{\vec{k},\lambda}\rangle$$

$$\dot{C}_{2,0}(t) = -i\sum_{\vec{k},\lambda} g_{\vec{k},\lambda} e^{i(\omega_{21}-\omega_{\vec{k}})t} C_{1,1_{\vec{k},\lambda}}(t)$$

$$\dot{C}_{1,1_{\vec{k},\lambda}}(t) = -i g^*_{\vec{k},\lambda} e^{-i(\omega_{21}-\omega_{\vec{k}})t} C_{2,0}(t)$$

infinite # of these

Formal Solution:

$$C_{1,1_{\vec{k},\lambda}}(t) = -i g^*_{\vec{k},\lambda} \int_0^t e^{-i(\omega_{21}-\omega_{\vec{k}})t'} C_{2,0}(t')\,dt'$$

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Expand

$$|\psi(t)\rangle = c_{2,0}(t)|2,0\rangle + \sum_{\vec{k},\lambda} c_{1,1_{\vec{k},\lambda}}(t)|1,1_{\vec{k},\lambda}\rangle$$

$$\dot{c}_{2,0}(t) = -i\sum_{\vec{k},\lambda} g_{\vec{k},\lambda} e^{i(\omega_{eg}-\omega_{\vec{k}})t} c_{1,1_{\vec{k},\lambda}}(t)$$

$$\dot{c}_{1,1_{\vec{k},\lambda}}(t) = -i\, g^*_{\vec{k},\lambda} e^{-i(\omega_{eg}-\omega_{\vec{k}})t} c_{2,0}(t)$$

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$$c_{1,1_{\vec{k},\lambda}}(t) = -i\, g^*_{\vec{k},\lambda}\int_0^t e^{-i(\omega_{eg}-\omega_{\vec{k}})t'} c_{2,0}(t')\,dt'$$

$$\dot{c}_{2,0}(t) = -\sum_{\vec{k},\lambda} |g_{\vec{k},\lambda}|^2 \int_0^t e^{i(\omega_{eg}-\omega_{\vec{k}})(t-t')} c_{2,0}(t')\,dt'$$

Time Dep. Perturbation Theory:

Study short-time limit, $c_{2,0}(t) \sim 1$

⇨ Fermi's Golden Rule

No good for this problem!

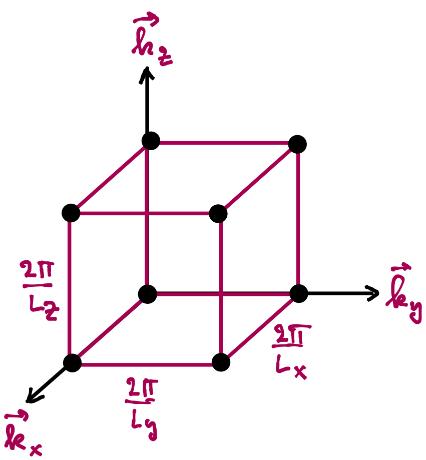
Wigner-Weisskopf Theory of Spontaneous Decay

$$\boxed{\dot{C}_{2,0}(t) = -\sum_{\vec{k},\lambda} |g_{\vec{k},\lambda}|^2 \int_0^t e^{i(\omega_{eg}-\omega_{\vec{k}})(t-t')} C_{2,0}(t')\,dt'}$$

Quantization in a box w/periodic B. C.

$\Rightarrow$ $k_i = n_i \frac{2\pi}{L}$, n **integer**

In $\vec{k}$ space the modes form a grid:



$$\frac{1 \text{ mode}}{\left(\frac{2\pi}{L_x}\right)\left(\frac{2\pi}{L_y}\right)\left(\frac{2\pi}{L_z}\right)} = \frac{V}{(2\pi)^3} = \mathcal{D}(\vec{k})$$

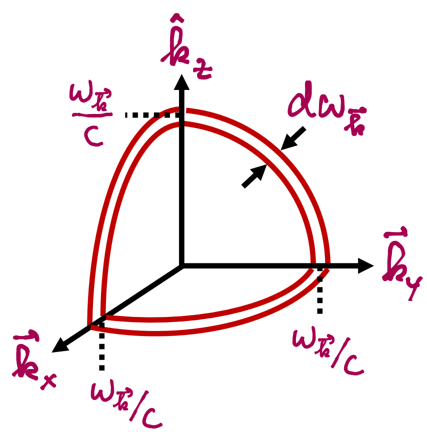
Density of Modes

Convert sum to integral over modes:

$$\sum_{\vec{k}} \to \int d^3\vec{k}\, \mathcal{D}(\vec{k}) = \int k^2 d(\hat{k})\, dk\, \mathcal{D}(\vec{k})$$

$$= \int d(\hat{k})\, d\omega_{\vec{k}} \frac{\omega_{\vec{k}}^2}{c^3} \mathcal{D}(\vec{k}) = \int d(\hat{k})\, d\omega_{\vec{k}}\, \mathcal{D}(\omega_{\vec{k}})$$

where $\hat{k} = \vec{k}/k$ is a unit vector along $\vec{k}$ and



$$\boxed{\mathcal{D}(\omega_{\vec{k}}) = \frac{V}{(2\pi)^3} \frac{\omega_{\vec{k}}^2}{c^3}}$$

mode density in shell of $\vec{k}$ - space of radius $\omega_{\vec{k}}/c$

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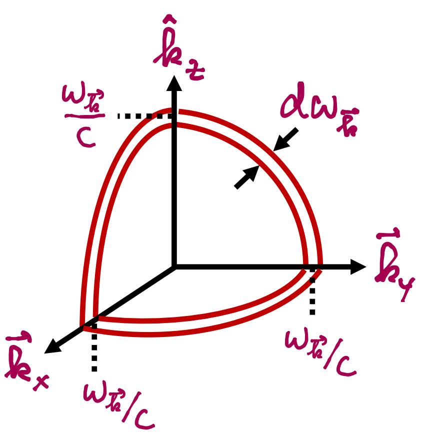

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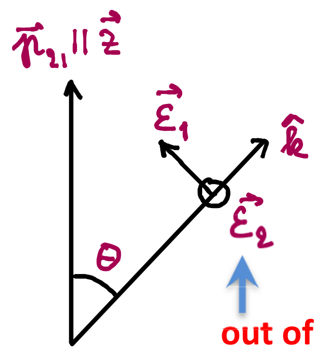
Thus, in the Continuum Limit

$$\sum_{\vec{k},\lambda} |g_{\vec{k},\lambda}|^2 \to \int_0^\infty d\omega_{\vec{k}}\, \mathcal{D}(\omega_{\vec{k}}) \sum_{\lambda} \int d(\hat{k})\, |g_{\vec{k},\lambda}|^2$$

We define the "polarization average"

$$\overline{|g(\omega_{\vec{k}})|^2} = \sum_{\lambda} \int d(\hat{k})\, |g_{\vec{k},\lambda}|^2 =$$

$$\frac{1}{\hbar^2} \underbrace{\left(\frac{\hbar\omega_{\vec{k}}}{2\varepsilon_0 V}\right)}_{\mathcal{E}_{\vec{k}}^2} \int d(\hat{k}) \sum_{\lambda} |\vec{p}_{21} \cdot \vec{\varepsilon}_{\vec{k},\lambda}|^2$$



out of plane

in polar coordinates

$$\sum_{\lambda} |\vec{p}_{21} \cdot \vec{\varepsilon}_{\vec{k},\lambda}|^2 = \sin^2\theta\, |\vec{p}_{12}|^2$$

no φ dependence

$$\int d(\hat{k}) \sum_{\lambda} |\vec{p}_{21} \cdot \vec{\varepsilon}_{\vec{k},\lambda}|^2 = \int_0^{2\pi} d\phi \int_0^1 d(\cos\theta) \sin^2\theta\, |\vec{p}_{21}|^2$$

$$= 2\pi |\vec{p}_{21}|^2 \int_{-1}^{1} du\,(1-u^2) = \boxed{\frac{8\pi}{3} |\vec{p}_{21}|^2}$$

Wigner-Weisskopf Theory of Spontaneous Decay

$$\boxed{\dot{c}_{2,0}(t) = -\boxed{\sum_{\vec{k},\lambda} |g_{\vec{k},\lambda}|^2} \int_0^t e^{i(\omega_{eg}-\omega_{\vec{k}})(t-t')} c_{2,0}(t')\,dt'}$$

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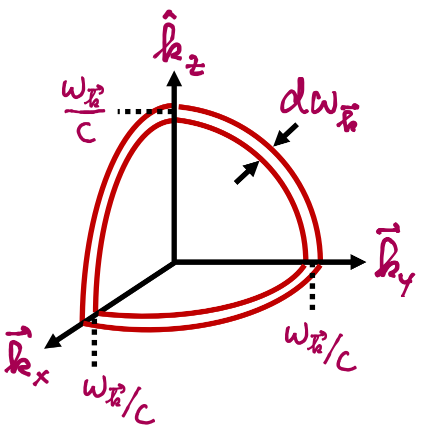
where $\hat{k} = \vec{k}/k$ **is a unit vector along** $\vec{k}$ **and**



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$$= \frac{4\pi\omega_{\vec{k}}}{2\hbar\varepsilon_0 V} |\vec{p}_{21}|^2$$

Putting it together:

$$\dot{c}_{2,0}(t) = -\sum_{\vec{k},\lambda} |g_{\vec{k},\lambda}|^2 \int_0^t e^{i(\omega_{21}-\omega_{\vec{k}})(t-t')} c_{2,0}(t')\,dt'$$

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Regular Perturbation Theory

Validity is limited to short times, such that the probability amplitude of the initial state does not change much. That allows us to approximate $c_{2,0}(t') \sim c_{2,0}(0)$. That does not work here.

Wigner-Weisskopf approximation:

The integral over time reduces to

$$\int_0^t dt'\, e^{i(\omega_{21}-\omega_{\vec{k}})(t-t')} \propto \delta(t-t')$$

on time scales $t-t' \gg |\omega_{21}-\omega_{\vec{k}}|^{-1}$

$$\dot{c}_{2,0}(t) = -\sum_{\vec{k},\lambda} |g_{\vec{k},\lambda}|^2 \int_0^t e^{i(\omega_{21}-\omega_{\vec{k}})(t-t')} c_{2,0}(t')\,dt'$$

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Wigner-Weisskopf Theory of Spontaneous Decay

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$$= -\int_0^\infty d\omega_{\vec{k}}\, |\overline{g(\omega_{\vec{k}})}|^2 \mathcal{D}(\omega_{\vec{k}}) \int_0^t dt' e^{i(\omega_{21}-\omega_{\vec{k}})(t-t')} c_{2,0}(t)$$

Rest of Lecture: Work with this Equation

First: This eq. is of the form $\dot{c}_{2,0}(t) = \beta c_{2,0}(t)$

$\Big\{$ solutions oscillate at freq. $\mathrm{Im}[\beta]$
and grow or decay at rate $\mathrm{Re}[\beta]$

Next: Atoms couple weakly to the vacuum

$c_{2,0}(t)$ changes slowly on timescale ω_{21}^{-1}, evolving at a rate Γ to be determined.

That means we can let $t \to \infty$ on timescale ω_{21}^{-1} while still keeping $t \ll \Gamma^{-1}$

Defining $-i\xi(\omega_{\vec{k}}-\omega_{21}) = \int_0^\infty dt' e^{i(\omega_{21}-\omega_{\vec{k}})(t-t')}$

We can then rewrite

$$\dot{c}_{2,0}(t) = -\int_0^\infty d\omega_{\vec{k}}\, |g(\omega_{\vec{k}})|^2 \mathcal{D}(\omega_{\vec{k}}) \left[-i\xi(\omega_{\vec{k}}-\omega_{21})\right] c_{2,0}(t)$$

Wigner-Weisskopf Theory of Spontaneous Decay

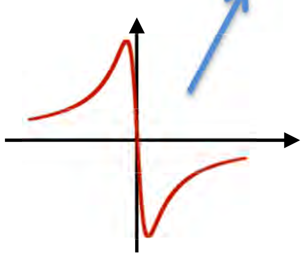
Math Details: Consider the function

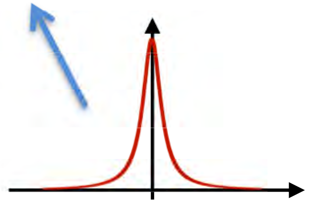
$$\xi(\Omega) = i\int_0^\infty d\tau\, e^{-i\Omega\tau}, \quad \Omega = \omega_{\vec{k}} - \omega_{21}$$

Approximate the integral as

$$\xi(\Omega) = \lim_{\varepsilon \to 0^+} i\int_0^\infty d\tau\, e^{-i\Omega\tau - \varepsilon\tau} = \lim_{\varepsilon \to 0^+}\left(\frac{i}{-i\Omega - \varepsilon}\right)$$

$$= \lim_{\varepsilon \to 0^+}\left(\frac{-\Omega^2}{\Omega^2+\varepsilon^2} + i\,\frac{\varepsilon}{\Omega^2+\varepsilon^2}\right)$$





$$= \mathbb{P}\left(\frac{1}{\Omega}\right) + i\pi\delta(\Omega)$$

Cauchy's Principal Part

Thus,

$$\dot{c}_{2,0}(t) \approx$$

$$-\left[\int_0^\infty d\omega_{\vec{k}}\, |g(\omega_{\vec{k}})|^2 \mathcal{D}(\omega_{\vec{k}})\left(\pi\delta(\omega_{21}-\omega_{\vec{k}}) - i\mathbb{P}\left(\frac{1}{\omega_{21}-\omega_{\vec{k}}}\right)\right)\right] c_{2,0}(t)$$

which is of the form

$$\boxed{\begin{aligned} \dot{c}_{2,0} &= -\left(\frac{A_{21}}{2} - i\delta\right) c_{2,0}(t) \\ c_{2,0}(t) &= e^{-A_{21}/2\, t}\, e^{i\delta(t)}\, c_{2,0}(0) \end{aligned}}$$

Decay Rate:

$$A_{21} = -\int_0^\infty d\omega_{\vec{k}}\, 2\pi\, |g(\omega_{\vec{k}})|^2 \mathcal{D}(\omega_{\vec{k}})\, \delta(\omega_{21}-\omega_{\vec{k}})$$

$$= 2\pi\, |g(\omega_{21})|^2 \mathcal{D}(\omega_{21})$$

$$= 2\pi \times \frac{4\pi\omega_{21}}{3\hbar\varepsilon_0 V} |\vec{p}_{21}|^2 \times \frac{V}{(2\pi)^3}\frac{\omega_{21}^2}{c^3}$$

Wigner-Weisskopf Theory of Spontaneous Decay

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$$\boxed{A_{21} = \frac{\omega_{21}^3 |\vec{p}_{21}|^2}{3\pi\varepsilon_0 \hbar c^3} = \frac{8\pi^2 |\vec{p}_{21}|^2}{3\pi\hbar\lambda^3}}$$

This is the value used earlier!

Frequency shift:

$$\delta = \lim_{\varepsilon\to 0}\left(i\int_0^\infty d\omega_{\vec{k}} |\overline{g(\omega_{\vec{k}})}|^2 \mathcal{D}(\omega_{\vec{k}}) \frac{\omega_{21}-\omega_{\vec{k}}}{(\omega_{21}-\omega_{\vec{k}})^2+\varepsilon^2}\right)$$

Integral is not well behaved!

Further development by Willis Lamb

Lamb shift

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Hydrogen 2P lifetime:

- accurate calculations for

$$\lambda = 121.57\,\text{nm}$$

$$|\vec{p}_{21}| = 0.745\, a_0 e = \frac{4\pi\varepsilon_0 \hbar^2}{m_e e^2} \times 0.745 e$$

$$\boxed{(A_{21})^{-1} = \tau_{2p} = 1.595\,\text{ns}}$$

Measured: $\tau_{2p} = 1.60\,\text{ns}$