# OPTI 544 Solution Set 6, Spring 2023

## Problem 1

**Note:** It is non-trivial to obtain the wave equation from the Lagrangian expressed in terms of the acoustic field $\eta(x)$, as this involves taking the derivative of a functional $(\mathcal{L})$ with respect to functions $(\eta(x),\dot{\eta}(x))$. We avoid the need to learn about functional derivatives by stating from the discrete Lagrangian,

$$\mathcal{L}=\frac{1}{2}\sum_i m\dot{x}_i^2-\kappa\left(x_{i+1}-x_i\right)^2=\frac{1}{2}\sum_i a\left[\frac{m}{a}\dot{x}_i^2-\kappa a\left(\frac{x_{i+1}-x_i}{a}\right)^2\right]$$

From the Lagrange equation of motion, $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_i}-\frac{\partial\mathcal{L}}{\partial x_i}=0$, we get an equation for each *i*:

$$\frac{m}{a}\ddot{x}_i-\frac{\kappa a}{a}\left(\frac{x_{i+1}-x_i}{a}\right)+\frac{\kappa a}{a}\left(\frac{x_{i+1}-x_i}{a}\right)=0 \qquad \text{Eq. (i)}$$

In the limit $a\to 0$ the last two terms become

$$\lim_{a\to 0}\left\{-\frac{Y}{a}\left[\left(\frac{\partial\eta}{\partial x}\right)_{x_i}-\left(\frac{\partial\eta}{\partial x}\right)_{x_i-a}\right]\right\}=-Y\frac{\partial^2\eta}{\partial x^2}$$

Also $$\lim_{a\to 0}\left\{\frac{m}{a}\ddot{x}_i\right\}=\mu\frac{\partial^2\eta}{\partial t^2}$$

Thus Eq. (i) above turns into a wave equation for the acoustic field,

$$\boxed{\mu\frac{\partial^2\eta}{\partial t^2}-Y\frac{\partial^2\eta}{\partial x^2}=0}$$

**Problem 2**

(a) Expressed in terms of the field $\eta(x)$, the kinetic energy is

$$T = \int dx \frac{1}{2}\mu\left(\frac{\eta(x,t)}{dt}\right)^2 = \sum_{k,k'}\frac{1}{2}\mu L\,\dot{q}_k\,\dot{q}_{k'}\int dx\, u_k(x)\, u_{k'}(x)$$
$$= \sum_{k,k'}\frac{1}{2}M\dot{q}_k^2$$

(b) Working out the expression for the potential energy is a bit more involved. First

$$V = \int dx \frac{1}{2}Y\left(\frac{\eta(x,t)}{dx}\right)^2 = \sum_{k,k'}\frac{1}{2}LY\, q_k\, q_{k'}\int dx\, \tfrac{d}{dx}u_k(x)\,\tfrac{d}{dx}u_{k'}(x) \tag{1}$$

Using integration by part, $\int f(x)G(x)\,dx = F(x)G(x) - \int F(x)g(x)\,dx$, we can rewrite the last part,

$$\int dx \frac{d}{dx}u_k(x)\frac{d}{dx}\,u_{k'}(x) = u_k(x)\frac{d}{dx}u_{k'}(x) - \int dx\; u_k(x)\frac{d^2}{dx^2}\,u_{k'}(x)$$

Substituting, we get

$$V = \sum_{k,k'}\frac{1}{2}LY\, q_k\, q_{k'}\left[u_k(x)\frac{d}{dx}u_{k'}(x) - \int dx\; u_k(x)\frac{d^2}{dx^2}\,u_{k'}(x)\right] \tag{2}$$

Next, we use

$$\sum_{k,k'}u_k(x)\frac{d}{dx}u_{k'}(x) = \frac{1}{2}\frac{d}{dx}\sum_{k,k'}u_k(x)u_{k'}(x) = \frac{1}{2}\frac{d^2}{dx^2}\sum_{k,k'}\int dx\; u_k(x)u_{k'}(x) = \frac{1}{2}\frac{d^2}{dx^2}\sum_{k,k'}\delta_{k,k'} = 0$$

where in the last step we have used $\int dx\; u_k(x)u_{k'}(x) = \delta_{kk'}$. Substituting in (2) we get

$$V = \sum_{k,k'}\frac{1}{2}LYq_k\, q_{k'}\left[\int dx\left(-\frac{d^2}{dx^2}u_k(x)\right)u_{k'}(x)\right] \tag{3}$$

Finally, we use

$$\frac{d^2}{dx^2}u_k(x) = -k^2 u_k(x) \Rightarrow \int dx \frac{d^2}{dx^2}u_k(x)u_{k'}(x) = k^2 \int dx u_k(x) u_{k'}(x) = k^2 \delta_{kk'},$$

where $k^2 = \frac{\omega_k^2}{v^2} = \frac{\mu}{Y}\omega_k^2$. Substituting in (3), we then get

$$V = \sum_{k,k'} \frac{1}{2} LY q_k q_{k'} k^2 \delta_{kk'} = \sum_k \frac{1}{2} LY \frac{\mu}{Y} \omega_k^2 q_k^2 = \sum_k \frac{1}{2} M\omega^2 q_k^2 \qquad (4)$$

This is the result given in the notes.

(c) The Lagrangian is

$$\mathcal{L} = T - V = \int dx \frac{1}{2}\mu\left(\frac{\eta(x,t)}{dt}\right)^2 - \int dx \frac{1}{2} Y \left(\frac{\eta(x,t)}{dx}\right)^2 = \sum_k \frac{1}{2} M \dot{q}_k^2 - \sum_k \frac{1}{2} M\omega^2 q_k^2$$

Plugging into the Lagrange equation of motion gives us

$$\frac{d}{dt}\frac{\partial \mathcal{L}_k}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}_k}{\partial q_k} = M\ddot{q}_k - M\omega^2 q_k^2 = 0 \Rightarrow \ddot{q}_k + \omega^2 q_k^2 = 0$$

This is the standard differential equation for a collection of harmonic oscillators, one for each normal mode *k*.