

OPTI 544 Solution Set 6

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Problem 1

Note: It is non-trivial to obtain the wave equation from the Lagrangian expressed in terms of the acoustic field $\eta(x)$, as this involves taking the derivative of a functional (\mathcal{L}) with respect to functions ($\eta(x), \dot{\eta}(x)$). We avoid the need to learn about functional derivatives by starting from the discrete Lagrangian,

$$\mathcal{L} = \frac{1}{2} \sum_i m \dot{x}_i^2 - \kappa (x_{i+1} - x_i)^2 = \frac{1}{2} \sum_i a \left[\frac{m}{a} \dot{x}_i^2 - \kappa a \left(\frac{x_{i+1} - x_i}{a} \right)^2 \right]$$

From the Lagrange equation of motion, $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} = 0$, we get an equation for each i :

$$\frac{m}{a} \ddot{x}_i - \frac{\kappa a}{a} \left(\frac{x_{i+1} - x_i}{a} \right) + \frac{\kappa a}{a} \left(\frac{x_{i+1} - x_i}{a} \right) = 0 \quad \text{Eq. (i)}$$

In the limit $a \rightarrow 0$ the last two terms become

$$\lim_{a \rightarrow 0} \left\{ -\frac{Y}{a} \left[\left(\frac{\partial \eta}{\partial x} \right)_{x_i} - \left(\frac{\partial \eta}{\partial x} \right)_{x_i - a} \right] \right\} = -Y \frac{\partial^2 \eta}{\partial x^2}$$

Also
$$\lim_{a \rightarrow 0} \left\{ \frac{m}{a} \ddot{x}_i \right\} = \mu \frac{\partial^2 \eta}{\partial t^2}$$

Thus Eq. (i) above turns into a wave equation for the acoustic field,

$$\mu \frac{\partial^2 \eta}{\partial t^2} - Y \frac{\partial^2 \eta}{\partial x^2} = 0$$

Problem 2

(a) Expressed in terms of the field $\eta(x)$, the kinetic energy is

$$\begin{aligned} T &= \int dx \frac{1}{2} \mu \left(\frac{\eta(x,t)}{dt} \right)^2 = \sum_{k,k'} \frac{1}{2} \mu L \dot{q}_k \dot{q}_{k'} \int dx u_k(x) u_{k'}(x) \\ &= \sum_{k,k'} \frac{1}{2} M \dot{q}_k^2 \end{aligned}$$

(b) Working out the expression for the potential energy is a bit more involved. First

$$V = \int dx \frac{1}{2} Y \left(\frac{\eta(x,t)}{dx} \right)^2 = \sum_{k,k'} \frac{1}{2} LY q_k q_{k'} \int dx \frac{d}{dx} u_k(x) \frac{d}{dx} u_{k'}(x) \quad (1)$$

Using integration by part, $\int f(x) G(x) dx = F(x) G(x) - \int F(x) g(x) dx$, we can rewrite the last part,

$$\int dx \frac{d}{dx} u_k(x) \frac{d}{dx} u_{k'}(x) = u_k(x) \frac{d}{dx} u_{k'}(x) - \int dx u_k(x) \frac{d^2}{dx^2} u_{k'}(x)$$

Substituting, we get

$$V = \sum_{k,k'} \frac{1}{2} LY q_k q_{k'} \left[u_k(x) \frac{d}{dx} u_{k'}(x) - \int dx u_k(x) \frac{d^2}{dx^2} u_{k'}(x) \right] \quad (2)$$

Next, we use

$$\sum_{k,k'} u_k(x) \frac{d}{dx} u_{k'}(x) = \frac{1}{2} \frac{d}{dx} \sum_{k,k'} u_k(x) u_{k'}(x) = \frac{1}{2} \frac{d^2}{dx^2} \sum_{k,k'} \int dx u_k(x) u_{k'}(x) = \frac{1}{2} \frac{d^2}{dx^2} \sum_{k,k'} \delta_{k,k'} = 0$$

where in the last step we have used $\int dx u_k(x) u_{k'}(x) = \delta_{kk'}$. Substituting in (2) we get

$$V = \sum_{k,k'} \frac{1}{2} LY q_k q_{k'} \left[\int dx \left(-\frac{d^2}{dx^2} u_k(x) \right) u_{k'}(x) \right] \quad (3)$$

Finally, we use

$$\frac{d^2}{dx^2} u_k(x) = -k^2 u_k(x) \Rightarrow \int dx \frac{d^2}{dx^2} u_k(x) u_{k'}(x) = k^2 \int dx u_k(x) u_{k'}(x) = k^2 \delta_{kk'},$$

where $k^2 = \frac{\omega_k^2}{v^2} = \frac{\mu}{Y} \omega_k^2$. Substituting in (3), we then get

$$V = \sum_{k,k'} \frac{1}{2} LY q_k q_{k'} k^2 \delta_{kk'} = \sum_k \frac{1}{2} LY \frac{\mu}{Y} \omega_k^2 q_k^2 = \sum_k \frac{1}{2} M \omega^2 q_k^2 \quad (4)$$

This is the result given in the notes.

(c) The Lagrangian is

$$\mathcal{L} = T - V = \int dx \frac{1}{2} \mu \left(\frac{\eta(x,t)}{dt} \right)^2 - \int dx \frac{1}{2} Y \left(\frac{\eta(x,t)}{dx} \right)^2 = \sum_k \frac{1}{2} M \dot{q}_k^2 - \sum_k \frac{1}{2} M \omega^2 q_k^2$$

Plugging into the Lagrange equation of motion gives us

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = M \ddot{q}_k - M \omega^2 q_k^2 = 0 \Rightarrow \ddot{q}_k + \omega^2 q_k^2 = 0$$

This is the standard differential equation for a collection of harmonic oscillators, one for each normal mode k .