# 1st Midterm 2024 - Problem Set

## I

OPTI 544 has explored optical physics in the context of several approximations, descriptions, and pictures. In the following, provide qualitative answers in your own words and keep math to an absolute minimum. No equations!

(a) Discuss the differences between fully classical, semiclassical and fully quantum descriptions of light, matter, and the interaction between the two. (8%)

(b) Under what conditions is it appropriate to use a fully classical description? (8%)

(c) OPTI 544 relies on the electric-dipole approximation from start to finish. State it in your own words. (7%)

(d) What is a Rotating Wave Approximation? When and how is it used? (9%)

(e) A semiclassical atom-field interaction can often be cast in terms of a state vector and a Schrödinger equation. What physical processes might make such a description insufficient? If so, what replaces state vectors and the Schrödinger equation in the description? (10%)

(f) We have mentioned in class that pulses of light can be used as mirrors and beam splitters for atomic matter waves. This is now commonly used to build Mach Zender interferometers in which single-atom wave packets are split, separated by several centimeters inside the interferometer, and recombined at the output. How can we reconcile this with part (c)? (8%)

## II

Consider an ensemble of identical two-level atoms, prepared such that at time $t = 0$ we have

$$\text{2/5 of the atoms in state } \left|\Psi_1(t=0)\right\rangle = \left|1\right\rangle$$

$$\text{3/5 of the atoms are in state } \left|\Psi_2(t=0)\right\rangle = \left|2\right\rangle$$

(a) Write out the density matrix $\rho(t=0)$ for the ensemble, in the basis $\{|1\rangle, |2\rangle\}$ of ground and excited states. Then perform one of the tests to find out if it is a pure state or a statistical mixture. (12%)

(b) Calculate the expectation value for the dipole moment operator $\langle \hat{p} \rangle$. Based on the result, do you expect the ensemble to absorb or emit light? (12%)

(c) Explain how you might have guessed the result in (b), given your understanding of the selection rules for electric dipole radiation. (13%)

(d) When driven and undergoing Rabi oscillations the states $\left|\Psi_1(t=0)\right\rangle$ and $\left|\Psi_2(t=0)\right\rangle$ will evolve into coherent superpositions of the states $|1\rangle$ and $|2\rangle$. Given what we know about electric dipole radiation, do you expect the ensemble to radiate light at later times? Explain your reasoning. (13%)

# 1st Midterm 2024 - Solution Set

**I**

(a) These are different levels of sophistication used to describe the interaction between light and matter.

Fully classical: The EM field is described by the classical Maxwells Equations.
Atoms are modeled as classical electron oscillators

Semiclassical: The EM field is described by the classical Maxwells Equations.
Atoms are described quantum mechanically.

Fully quantum: The EM field is described by the quantum Maxwells Equations.
Atoms are described quantum mechanically.

(b) A fully classical description is adequate do describe the interaction between the EM field and quantum 2-level atoms if the atomic excitation is negligible.

(c) The EDA assumes that the extent of the atom (quantum or classical) is far smaller than an optical wavelength.

(d) When solving the Rabi problem, we first write out the Schrödinger equation as a set of coupled, linear differential equations for the ground and excited state probability amplitudes. These probability amplitudes oscillate as $e^{+i\omega t}$ or $^{-i\omega t}$, where $\omega$ is the frequency of the driving field. The RWA begins with a change of variables such that each equation contains a set of slowly evolving probability amplitudes, and a second set that oscillates as $e^{\pm i2\omega t}$ and therefore average out over the relevant timescales. A similar transformation is often used in other situations where the equations of motion involves both slow and fast variables.

(e) If the initial state is fully known and the time evolution is perfectly governed by a Schrödinger equation, then the physical state is perfectly known at all later times and can be described by a state vector. However, if the initial state is not perfectly known or if the Schrödinger equation is noisy, then after some time we won't know the exact quantum state. In that situation we can describe the physical state as a probability distribution over quantum states, and we use a density operator/matrix to describe it.

(f) The EDA is concerned with the internal degree of freedom for the bound nuclear-electronic system. If the electron-nuclear separation is much less than an optical wavelength then the EDA is valid. The matter wave interferometry experiments are concerned with the center-of-mass degree of freedom for the bound nuclear-electronic system. Thus, inside the interferometry the bound system is in a superposition state of wave packets traveling along macroscopically distinct paths, and the electron-nuclear separation is not affected.

**II**

Consider an ensemble of identical two-level atoms, prepared such that at time $t=0$ we have

$$\text{2/5 of the atoms are in state } |\Psi_1(t=0)\rangle = |1\rangle$$

$$\text{3/5 of the atoms are in state } |\Psi_2(t=0)\rangle = |2\rangle$$

(a) We can find the density matrix associated with this ensemble by noting that

$$|\Psi_1(t=0)\rangle = |1\rangle \text{ corresponds to the density operator } \quad \rho_1 = |1\rangle\langle 1|$$

$$|\Psi_1(t=0)\rangle = |2\rangle \text{ corresponds to the density operator } \quad \rho_2 = |2\rangle\langle 2|$$

In matrix form we have $\rho_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\rho_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

The density matrix for the ensemble is thus

$$\rho = \frac{2}{5}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{3}{5}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2/5 & 0 \\ 0 & 3/5 \end{pmatrix}$$

Check trace: $\mathrm{Tr}[\rho] = \begin{pmatrix} 2/5 & 0 \\ 0 & 3/5 \end{pmatrix} = 1$

Check purity: $\rho^2 = \begin{pmatrix} 2/5 & 0 \\ 0 & 3/5 \end{pmatrix}^2 = \begin{pmatrix} 4/25 & 0 \\ 0 & 9/25 \end{pmatrix}$

and $Tr[\rho^2] = Tr\left[\begin{pmatrix} 2/5 & 0 \\ 0 & 3/5 \end{pmatrix}^2\right] = Tr\begin{pmatrix} 4/25 & 0 \\ 0 & 9/25 \end{pmatrix} = \frac{13}{25} < 1$

Therefore the state must be a statistical mixture (mixed.)

(b) The expectation value for the ensemble averaged dipole moment is

$$\mathrm{Tr}[\rho\hat{\vec{p}}] = \mathrm{Tr}\left[\begin{pmatrix} 2/5 & 0 \\ 0 & 3/5 \end{pmatrix}\begin{pmatrix} 0 & \vec{p}_{12} \\ \vec{p}_{21} & 0 \end{pmatrix}\right] = \mathrm{Tr}\left[\begin{pmatrix} 0 & \frac{2}{5}\vec{p}_{12} \\ \frac{3}{5}\vec{p}_{21} & 0 \end{pmatrix}\right] = 0$$

(c) Dipole radiation is associated with non-zero off-diagonal elements in the density matrix. These are zero in the case above, so no radiation is emitted by the ensemble.

We could have guessed that easily, by noting that we are dealing with two entirely separate sets of atoms, 2/5th or which are in the ground state and 3/5ths in the excited state. Neither set of atoms can radiate due to the parity selection rule that forbids radiation from atoms in parity eigenstates.

(d) Each sub-ensemble is incapable of radiating while the atoms are in parity eigenstates. When the sub-ensembles undergoes Rabi oscillations, driven either by the same or different light fields, the initial states will evolve into superpositions of the states $|1\rangle$ and $|2\rangle$. As we know, radiation is strongest when atoms are in an equal superposition of ground and excited states. Given the initial admixtures of ground and excited state in the two sub-ensembles, radiation from the more numerous atoms is guaranteed to dominate at least some of the time.

Postscript: A number of you have correctly pointed out that during Rabi oscillation the ensemble will radiate light due to spontaneous decay of atoms in the excited state. In the regime where the Rabi frequency is much larger than the spontaneous decay rate, $\chi >> A_{21}$, coherent radiation averaged over the Rabi cycle typically dominates over spontaneous emission. That was what was in my mind when I put together question II(d).