Quantum Electrodynamics – QED

Starting point: Maxwells Equations

(1) $\nabla \cdot \vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0}\rho(\vec{r},t)$

(2) $\nabla \cdot \vec{B}(\vec{r},t) = 0$

(3) $\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t)$

(4) $\nabla \times \vec{B}(\vec{r},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{r},t) + \frac{1}{\varepsilon_0 c^2}\vec{j}(\vec{r},t)$

Implicit: Charges & Fields in Vacuum
No "medium response"

Same issue as with our introductory example:

Maxwells eqs are non-local

We need to put the classical description in proper form -> Normal Mode expansion

Free Fields - Switch to Fourier Domain

(1) $i\vec{k} \cdot \vec{\mathcal{E}}(\vec{k},t) = \frac{1}{\varepsilon_0}\rho(\vec{k},t)$

(2) $i\vec{k} \cdot \vec{\mathcal{B}}(\vec{k},t) = 0$

(3) $i\vec{k} \times \vec{\mathcal{E}}(\vec{k},t) = -\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t)$

(4) $i\vec{k} \times \vec{\mathcal{B}}(\vec{k},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathcal{E}}(\vec{k},t) + \frac{1}{\varepsilon_0 c^2}\vec{\mathcal{J}}(\vec{k},t)$

Fourier Transform:

$$\left\{ \begin{array}{l} \nabla \cdot \vec{G} \rightleftarrows i\vec{k} \cdot \vec{\mathcal{G}} \\ \nabla \times \vec{G} \rightleftarrows i\vec{k} \times \vec{\mathcal{G}} \end{array} \right.$$

<u>Note</u>: This is a Normal Mode decomposition

No charges -> No coupling between modes with different $\vec{k}$

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Free Fields - Switch to Fourier Domain

(1) $i\vec{k}\cdot\vec{\mathcal{E}}(\vec{k},t)=\frac{1}{\varepsilon_0}\rho(\vec{k},t)$

(2) $i\vec{k}\cdot\vec{\mathcal{B}}(\vec{k},t)=0$

(3) $i\vec{k}\times\vec{\mathcal{E}}(\vec{k},t)=-\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t)$

(4) $i\vec{k}\times\vec{\mathcal{B}}(\vec{k},t)=\frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathcal{E}}(\vec{k},t)+\frac{1}{\varepsilon_0 c^2}\vec{j}(\vec{k},t)$

Fourier Transform:

$$\begin{cases}\nabla\cdot\vec{G} \rightleftarrows i\vec{k}\cdot\vec{g}\\ \nabla\times\vec{G} \rightleftarrows i\vec{k}\times\vec{g}\end{cases}$$

<u>Note</u>: This is a Normal Mode decomposition

No charges -> No coupling between modes with different $\vec{k}$

Separate into Transverse & Longitudinal Fields

$$\vec{\mathcal{E}}(\vec{k},t)=\vec{\mathcal{E}}_\parallel(\vec{k},t)+\vec{\mathcal{E}}_\perp(\vec{k},t)$$

$$\vec{\mathcal{B}}(\vec{k},t)=\cancel{\vec{\mathcal{B}}_\parallel(\vec{k},t)}+\vec{\mathcal{B}}_\perp(\vec{k},t)\quad \text{MEq (2)}$$

Entirely Transverse

<u>Note</u>:
- $\vec{\mathcal{E}}_\parallel$ is $\frac{\vec{k}}{k}$ × the projection of $\vec{\mathcal{E}}$ onto $\vec{k}$
- $\mathcal{E}_\parallel=-\frac{i}{k}\,i\vec{k}\cdot\vec{\mathcal{E}}$ is the projection of $\vec{\mathcal{E}}$ onto $\vec{k}$

$$\vec{\mathcal{E}}_\parallel=\frac{\vec{k}}{k}\mathcal{E}_\parallel=\frac{\vec{k}}{k}\left(-\frac{i}{k}\,i\vec{k}\cdot\vec{\mathcal{E}}\right)=\frac{\vec{k}}{\varepsilon_0 k^2}\rho(\vec{k},t)$$

Coulomb field from the charges

Only $\vec{\mathcal{E}}_\perp$ and $\vec{\mathcal{B}}_\perp$ are new degrees of freedom beyond the particles -> <u>Free Fields</u>

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Free Fields - Switch to Fourier Domain

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Fourier Transform: $\begin{cases} \nabla\cdot\vec{G} \rightleftarrows i\vec{k}\cdot\vec{\mathcal{G}} \\ \nabla\times\vec{G} \rightleftarrows i\vec{k}\times\vec{\mathcal{G}} \end{cases}$

Note: This is a Normal Mode decomposition

No charges -> No coupling between modes with different $\vec{k}$

Separate into Transverse & Longitudinal Fields

$$\vec{\mathcal{E}}(\vec{k},t) = \vec{\mathcal{E}}_\parallel(\vec{k},t) + \vec{\mathcal{E}}_\perp(\vec{k},t)$$

$$\vec{\mathcal{B}}(\vec{k},t) = \cancel{\vec{\mathcal{B}}_\parallel(\vec{k},t)} + \vec{\mathcal{B}}_\perp(\vec{k},t) \quad \text{MEq (2)}$$

Entirely Transverse

Note: $\begin{cases} \vec{\mathcal{E}}_\parallel \text{ is } \frac{\vec{k}}{k} \times \text{ the projection of } \vec{\mathcal{E}} \text{ onto } \vec{k} \\ \mathcal{E}_\parallel = -\frac{i}{k} i\vec{k}\cdot\vec{\mathcal{E}} \text{ is the projection of } \vec{\mathcal{E}} \text{ onto } \vec{k} \end{cases}$

$$\vec{\mathcal{E}}_\parallel = \frac{\vec{k}}{k}\mathcal{E}_\parallel = \frac{\vec{k}}{k}\left(-\frac{i}{k} i\vec{k}\cdot\vec{\mathcal{E}}\right) = \frac{\vec{k}}{\varepsilon_0 k^2}\rho(\vec{k},t)$$

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Entirely Transverse

Note:
- $\vec{\mathcal{E}}_{\parallel}$ is $\frac{\vec{k}}{k} \times$ the projection of $\vec{\mathcal{E}}$ onto $\vec{k}$
- $\mathcal{E}_{\parallel} = -\frac{i}{k} i \vec{k} \cdot \vec{\mathcal{E}}$ is the projection of $\vec{\mathcal{E}}$ onto $\vec{k}$

MEq (1)

$$\vec{\mathcal{E}}_{\parallel} = \frac{\vec{k}}{k}\mathcal{E}_{\parallel} = \frac{\vec{k}}{k}\left(-\frac{i}{k} i \vec{k}\cdot\vec{\mathcal{E}}\right) = \frac{\vec{k}}{\varepsilon_0 k^2}\rho(\vec{k},t)$$

Coulomb field from the charges

Only $\vec{\mathcal{E}}_{\perp}$ and $\vec{\mathcal{B}}_{\perp}$ are new degrees of freedom beyond the particles -> Free Fields

Eqs for Transverse Fields, from MEqs (3) & (4)

(5a) $$\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t) = -i\vec{k} \times \vec{\mathcal{E}}_{\perp}(\vec{k},t)$$

(6a) $$\frac{\partial}{\partial t}\vec{\mathcal{E}}_{\perp}(\vec{k},t) = c^2 i\vec{k} \times \vec{\mathcal{B}}(\vec{k},t) - \frac{1}{\varepsilon_0}\vec{j}_{\perp}(\vec{k},t)$$

inverse FT

(5b) $$\frac{\partial}{\partial t}\vec{B}(\vec{r},t) = -\nabla \times \vec{E}_{\perp}(\vec{r},t)$$

(6b) $$\frac{\partial}{\partial t}\vec{E}_{\perp}(\vec{r},t) = c^2 \nabla \times \vec{B}(\vec{r},t) - \frac{1}{\varepsilon_0}\vec{j}_{\perp}(\vec{r},t)$$

combine (5b) & (6b)

Wave Equation for the Free Fields

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_{\perp}(\vec{r},t) = \frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t}\vec{j}_{\perp}(\vec{r},t)$$

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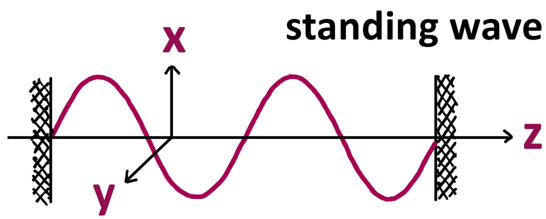
Normal Modes in a 1D Cavity

Length L

Cross section A

Volume $V = LA$



Normal Modes are Standing Waves

Let $\vec{E}(z,t) = \vec{\mathcal{E}}_x E_x(z,t)$ and expand

fiducial mass

(7) $E_x(z,t) = \sum_j A_j q_j(t) \sin(k_j z)$, $A_j = \sqrt{\frac{2\omega_j m_j}{\varepsilon_0 V}}$

Generalized coordinate

MEq (4) w/no charges

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}_\perp(\vec{r},t) = \vec{\mathcal{E}}_x \boxed{\frac{1}{c^2}\sum_j A_j \dot{q}_j(t) \sin(k_j t)}$$

$$= \vec{\mathcal{E}}_x \left(\cancel{\frac{\partial B_z}{\partial y}} - \frac{\partial B_y}{\partial z} \right) = -\vec{\mathcal{E}}_x \boxed{\frac{\partial B_y}{\partial z}}$$

$\mathcal{B}$ transverse ⇒ $B_z = 0$

From Eq. (5a) we see that

$$\vec{B} \perp \vec{E}, \vec{\mathcal{E}}_z \Rightarrow \vec{B}(z,t) = \vec{\mathcal{E}}_y B_y(z,t)$$

Putting this together we get

$$\frac{\partial B_y}{\partial z} = -\sum_j \frac{A_j}{c^2} q_j(t) \sin(k_j z)$$

(8) $B_y(z,t) = \sum_j \frac{A_j}{k_j c^2} \dot{q}_j(t) \cos(k_j z)$

Generalized velocity

Hamiltonian (Energy) for the Classical Field

$$\mathcal{H} = \frac{\varepsilon_0 A}{2}\int_0^L dz \left(|\vec{E}|^2 + c^2 |\vec{B}|^2 \right) =$$

$$\frac{\varepsilon_0 A}{2}\int_0^L dz \sum_j \left[A_j^2 q_j(t)^2 \sin^2(k_j z) + \frac{A_j^2}{k_j} \dot{q}_j(t)^2 \cos^2(k_j z) \right]$$

Quantum Electrodynamics – QED

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Generalized velocity

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$$\frac{\varepsilon_0 A}{2} \int_0^L dz \sum_j \left[A_j^2 q_j(t)^2 \sin^2(k_j z) + \frac{A_j^2}{k_j} \dot{q}_j(t)^2 \cos^2(k_j z) \right]$$

Integrating over the Cavity volume

$$A \int_0^L dz \sin^2(k_j z) = A \int_0^L dz \cos^2(k_j z) = V/2$$

and substituting $A_j^2 = \frac{2\omega_j^2 m_j}{\varepsilon_0 V}$ we finally get

$$\boxed{\mathcal{H} = \sum_j \left[\frac{1}{2} m_j \omega_j^2 q_j^2 + \frac{1}{2} m_j \dot{q}_j^2 \right]}$$

Lagrangian for the Classical Field

$$\mathcal{L} = \frac{\varepsilon_0 A}{2} \int_0^L dz \left(c^2 |\vec{B}|^2 - |\vec{E}|^2 \right) \quad ☑$$

$$= \sum_j \left[\frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j \omega_j^2 q_j^2 \right]$$

Check $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E}_\perp(\vec{r}, t) = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$$

Quantum Electrodynamics – QED

Integrating over the Cavity volume

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$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_\perp(\vec{r},t) = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

And Finally:

Conjugate Momentum $\boxed{p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = m_j \dot{q}_j}$

As before, a collection of Harmonic Oscillators, ready for quantization!

Quantum Electrodynamics – QED

To summarize so far…

$$\boxed{\begin{aligned} E_x(z,t) &= \sum_j A_j q_j(t)\sin(k_j z) \qquad A_j = \sqrt{\frac{\omega_j^2 m_j}{2\varepsilon_0 V}} \\ B_y(z,t) &= \sum_j \frac{A_j}{k_j c^2}\dot{q}_j(t)\cos(k_j z) \end{aligned}}$$

$$\boxed{\begin{aligned} \mathcal{L} &= \frac{\varepsilon_0 A}{2}\int_0^L dz\left(c^2|\vec{B}|^2 - |\vec{E}|^2\right) \\ &= \sum_j\left[\frac{1}{2}m_j\dot{q}_j^2 - \frac{1}{2}m_j q_j^2\right] \end{aligned}}$$

$$\boxed{p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = m_j \dot{q}_j}$$

$$\boxed{\mathcal{H} = \sum_j\left[\frac{1}{2}m_j\dot{q}_j^2 - \frac{1}{2}m_j\omega_j^2 q_j^2\right]}$$

Classical Fields

Dimensionless Field Variables:

$$Q_j = q_j/q_{o,j}\,,\quad q_{o,j} = \sqrt{2\hbar/m_j\omega_j}$$

$$P_j = p_j/p_{o,j}\,,\quad p_{o,j} = \sqrt{2\hbar m_j\omega_j}$$

$$\boxed{\alpha_j(t) = Q_j(t) + i\,P_j(t) = \alpha_j(0)e^{-i\omega_j t}}$$

$$\boxed{\begin{aligned} E_x(z,t) &= \sum_j A_j q_j(t)\sin(k_j z)\,, \qquad \mathcal{E}_j = A_j q_{o,j} = \sqrt{\frac{\hbar\omega_j}{\varepsilon_0 V}} \\ &= \sum_j \mathcal{E}_j\left[\alpha_j(t) + \alpha_j^*(t)\right]\sin(k_j z) \\ B_y(z,t) &= \sum_j \frac{A_j}{k_j c^2}\dot{q}_j(t)\cos(k_j z) \\ &= -\frac{i}{c}\sum_j \mathcal{E}_j\left[\alpha_j(t) - \alpha_j^*(t)\right]\cos(k_j t) \end{aligned}}$$

field
"per photon"

Quantum Electrodynamics – QED

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$$Q_j = q_j / q_{0,j}\,, \quad q_{0,j} = \sqrt{2\hbar / m_j \omega_j}$$

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$$E_x(z,t) = \sum_j A_j\, q_j(t) \sin(k_j z)\,, \qquad \mathcal{E}_j = A_j\, q_{0,j} = \sqrt{\frac{\hbar \omega_j}{\varepsilon_0 V}}$$

$$= \sum_j \mathcal{E}_j \left[\alpha_j(t) + \alpha_j^*(t)\right] \sin(k_j z)$$

field "per photon"

$$B_y(z,t) = \sum_j \frac{A_j}{k_j c^2}\, \dot{q}_j(t) \cos(k_j z)$$

$$= -\frac{i}{c} \sum_j \mathcal{E}_j \left[\alpha_j(t) - \alpha_j^*(t)\right] \cos(k_j t)$$

Standard Quantization Procedure

$$q_j \to \hat{q}_j\,, \qquad p_j \to \hat{p}_j\,, \qquad [\hat{q}_j, \hat{p}_{j'}] = i\hbar\, \delta_{jj'}$$

$$\alpha_j(t) \to \hat{a}_j\,, \qquad \alpha_j^*(t) \to \hat{a}_j^+\,, \qquad [\hat{a}_j, \hat{a}_{j'}^+] = \delta_{jj'}$$

$$\hat{E}_x(z) = \sum_j \mathcal{E}_j \left(\hat{a}_j + \hat{a}_j^+\right) \sin(k_j z)$$

$$\hat{B}_y(z) = -\frac{i}{c} \sum_j \mathcal{E}_j \left(\hat{a}_j - \hat{a}_j^+\right) \cos(k_j z)$$

Total Field

$$\hat{\vec{E}}(z) = \vec{\varepsilon}_x\, \hat{E}_x(z) + \vec{\varepsilon}_y\, \hat{E}_y(z)$$

$$\hat{\vec{B}}(z) = \vec{\varepsilon}_x\, \hat{B}_x(z) + \vec{\varepsilon}_y\, \hat{B}_y(z)$$

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Note:

These are the Field Operators in the Schrödinger Picture (*t*-dependence in states)

Often advantageous to use Heisenberg Picture (*t*-dependence in operators)

$$\alpha_j(t) \to \hat{a}_j(t) = \hat{a}_j(0)e^{-i\omega_j t}$$

Field Quantization in Free Space:

Normal Modes : $\vec{u}_{\vec{k},\lambda}(\vec{r}) = \vec{\mathcal{E}}_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k}} t - \vec{k}\cdot\vec{r})} + c.c.$

λ: polarization index

Finite quantization volume: $\mathcal{E}_{\vec{k}} = \sqrt{\hbar\omega_j / 2\varepsilon_0 V}$

$L \times L \times L$

L large -> nature of boundary conditions not important

Periodic boundary conditions

$$|\vec{k}| = n\,2\pi/L$$

Quantum Electrodynamics – QED

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Classical Fields (Fourier Sum):

$$\vec{E}_\perp(\vec{r},t) = \sum_{\vec{k},\lambda} \vec{\varepsilon}_{\vec{k},\lambda}\, \mathcal{E}_{\vec{k},\lambda}\, \alpha_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k}} t - \vec{k}\cdot\vec{r})} + c.c.$$

$$\vec{B}_\perp(\vec{r},t) = \sum_{\vec{k},\lambda} \frac{\vec{k}\times\vec{\varepsilon}_{\vec{k},\lambda}}{kc}\, \mathcal{E}_{\vec{k},\lambda}\, \alpha_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k}} t - \vec{k}\cdot\vec{r})} + c.c.$$

Quantization:

$$\alpha_{\vec{k},\lambda} \rightarrow \hat{a}_{\vec{k},\lambda}, \quad [\hat{a}_{\vec{k},\lambda}, \hat{a}^+_{\vec{k}',\lambda'}] = \delta_{\vec{k},\vec{k}'}\delta_{\lambda,\lambda'}$$

$$\alpha^*_{\vec{k},\lambda} \rightarrow \hat{a}^+_{\vec{k},\lambda}, \quad [\hat{a}_{\vec{k},\lambda}, \hat{a}_{\vec{k}',\lambda'}] = [\hat{a}^+_{\vec{k},\lambda}, \hat{a}^+_{\vec{k}',\lambda'}] = 0$$

$$\hat{\vec{E}}_\perp(\vec{r},t) = \sum_{\vec{k},\lambda} \vec{\varepsilon}_{\vec{k},\lambda}\, \mathcal{E}_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k},\lambda} t - \vec{k}\cdot\vec{r})} + H.C.$$

$$\hat{\vec{B}}(\vec{r},t) = \sum_{\vec{k},\lambda} \frac{\vec{k}\times\vec{\varepsilon}_{\vec{k},\lambda}}{kc}\, \mathcal{E}_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k},\lambda} t - \vec{k}\cdot\vec{r})} + H.C.$$

– Heisenberg Picture –

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Quantization:

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– Heisenberg Picture –

Positive & Negative Frequency Components:

$$\hat{\vec{E}}_\perp(\vec{r},t) = \hat{\vec{E}}^{(+)}(\vec{r},t) + \hat{\vec{E}}^{(-)}(\vec{r},t)$$

$$\hat{\vec{E}}^{(+)}(\vec{r},t) = \sum_{\vec{k},\lambda} i\vec{\varepsilon}_{\vec{k},\lambda}\, \mathcal{E}_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda}\, e^{i(\vec{k}\cdot\vec{r} - \omega_{\vec{k},\lambda} t)}$$

$$\hat{\vec{E}}^{(-)}(\vec{r},t) = \sum_{\vec{k},\lambda} -i\vec{\varepsilon}^{\,*}_{\vec{k},\lambda}\, \mathcal{E}^{*}_{\vec{k},\lambda}\, \hat{a}^{+}_{\vec{k},\lambda}\, e^{-i(\vec{k}\cdot\vec{r} - \omega_{\vec{k},\lambda} t)}$$

Quantum Electrodynamics – QED

Classical Fields (Fourier Sum):

$$\vec{E}_\perp(\vec{r},t) = \sum_{\vec{k},\lambda} \vec{\varepsilon}_{\vec{k},\lambda}\, \mathcal{E}_{\vec{k},\lambda}\, \alpha_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k}} t - \vec{k}\cdot\vec{r})} + c.c.$$

$$\vec{B}_\perp(\vec{r},t) = \sum_{\vec{k},\lambda} \frac{\vec{k}\times\vec{\varepsilon}_{\vec{k},\lambda}}{kc}\, \mathcal{E}_{\vec{k},\lambda}\, \alpha_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k}} t - \vec{k}\cdot\vec{r})} + c.c.$$

Quantization:

$$\alpha_{\vec{k},\lambda} \to \hat{a}_{\vec{k},\lambda}, \qquad [\hat{a}_{\vec{k},\lambda}, \hat{a}^+_{\vec{k}',\lambda'}] = \delta_{\vec{k},\vec{k}'}\delta_{\lambda,\lambda'}$$

$$\alpha^*_{\vec{k},\lambda} \to \hat{a}^+_{\vec{k},\lambda}, \qquad [\hat{a}_{\vec{k},\lambda}, \hat{a}_{\vec{k}',\lambda'}] = [\hat{a}^+_{\vec{k},\lambda}, \hat{a}^+_{\vec{k}',\lambda'}] = 0$$

$$\hat{\vec{E}}_\perp(\vec{r},t) = \sum_{\vec{k},\lambda} \vec{\varepsilon}_{\vec{k},\lambda}\, \mathcal{E}_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k},\lambda} t - \vec{k}\cdot\vec{r})} + H.C.$$

$$\hat{\vec{B}}(\vec{r},t) = \sum_{\vec{k},\lambda} \frac{\vec{k}\times\vec{\varepsilon}_{\vec{k},\lambda}}{kc}\, \mathcal{E}_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k},\lambda} t - \vec{k}\cdot\vec{r})} + H.C.$$

– Heisenberg Picture –

Comment on Quantization Procedure:

Even after a choice of normal modes has been made there is no unique way of quantizing the electromagnetic field. In our treatment we have chosen to associate a normal coordinate q_i with the electric field and a conjugate momentum with the magnetic field in mode j. It is also common to work with the vector potential $\vec{A}$ rather than the fields $\vec{E}$ and $\vec{B}$. In the *Coulomb gauge* the transverse vector field $\vec{A}_\perp$ describes the free fields , and one typically uses $\vec{A}_{\perp j}$ and $\dot{A}_{\perp j}$ as coordinate and conjugate momenta for the field in mode j. Note that this associates the normal coordinate with the magnetic field and the conjugate momentum with the electric field. As a consequence, in the equations for the fields one will get $E_x \propto (\hat{a}_j - \hat{a}_j^\dagger)$ and $B_y \propto (\hat{a}_j + \hat{a}_j^\dagger)$. While confusing, these are merely different ways of putting together the formalism, as is so commonly seen in quantum mechanics. In the end, all predictions for physically mesurable quantities are independent of such internal differences in the formalism.

Quantum Electrodynamics – QED

Positive & Negative Frequency Components:

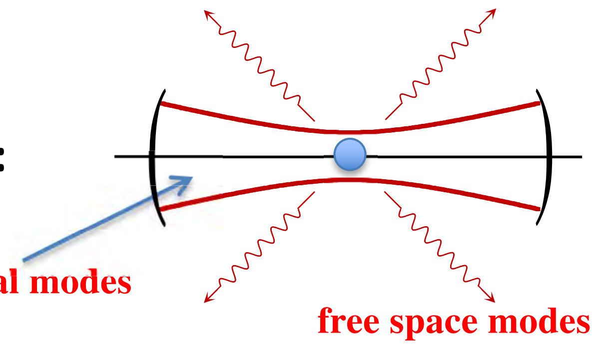
$$\hat{\vec{E}}_{\perp}(\vec{r},t) = \hat{\vec{E}}^{(+)}(\vec{r},t) + \hat{\vec{E}}^{(-)}(\vec{r},t)$$

$$\hat{\vec{E}}^{(+)}(\vec{r},t) = \sum_{\vec{k},\lambda} i\vec{\varepsilon}_{\vec{k},\lambda}\,\mathcal{E}_{\vec{k},\lambda}\,\hat{a}_{\vec{k},\lambda}\,e^{i(\vec{k}\cdot\vec{r}-\omega_{\vec{k},\lambda}t)}$$

$$\hat{\vec{E}}^{(-)}(\vec{r},t) = \sum_{\vec{k},\lambda} -i\vec{\varepsilon}^{\,*}_{\vec{k},\lambda}\,\mathcal{E}^{*}_{\vec{k},\lambda}\,\hat{a}^{+}_{\vec{k},\lambda}\,e^{-i(\vec{k}\cdot\vec{r}-\omega_{\vec{k},\lambda}t)}$$

Other Normal Modes Sets

Atom in Cavity:



Wavepackets: (Milloni & Eberly, Sec. 12.8, p 381) (QED lecture notes, p 16)

Classical field pulse envelope

$$\vec{E}(\vec{r},t) = \vec{\varepsilon}\,\mathcal{E}_0\,u(z-ct)\,e^{i(k_0 z-\omega_0 t)} + c.c.$$

Mode volume $V = \int d^3r\,|u(x,y,z-ct)|^2$

Quantization $\mathcal{E}_0 \rightarrow \mathcal{E}_k \alpha_k \rightarrow \mathcal{E}_k \hat{a}_k$ etc.

Wave-Particle Duality similar for Photons and Phonons