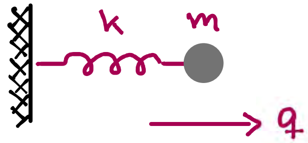
Quantum Electrodynamics – Intro to Field Theory

Classical Simple Harmonic Oscillator (SHO)

Particle on a spring



$\omega = \sqrt{k/m}$

osc. freq.

dynamical coordinate.

Kinetic Energy: $T = \frac{1}{2} m \dot{q}^2$

Potential Energy: $V = \frac{1}{2} k q^2 = \frac{1}{2} m \omega^2 q^2$

Lagrangian: $\mathcal{L} = T - V = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2$

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0 \Rightarrow \ddot{q} + \omega^2 q = 0$$

usual eq. of motion

Conjugate momentum $p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m \dot{q}$

Hamiltonian

$$\mathcal{H} = T(\dot{q} = p/m) + V(q) = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2$$

$$\left.\begin{aligned} \dot{q} &= \frac{\partial \mathcal{H}}{\partial p} = p/m \\ \dot{p} &= -\frac{\partial \mathcal{H}}{\partial q} = -m\omega^2 q \end{aligned}\right\} \Rightarrow \ddot{q} + \omega^2 q = 0$$

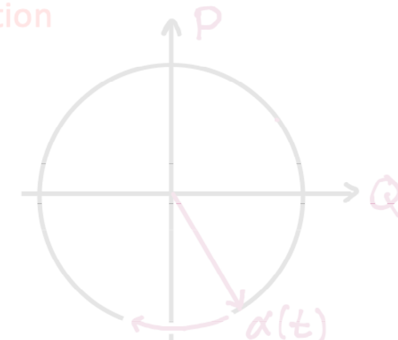
solution

Scaled variables

$Q \equiv q/q_0, \quad P = p/p_0$

$$\alpha = Q + iP \begin{cases} Q = \mathrm{Re}[\alpha] \\ P = \mathrm{Im}[\alpha] \\ \mathcal{H} = E_0 \alpha^* \alpha \end{cases}$$

Phase plane



Quantum Electrodynamics – Intro to Field Theory

Conjugate momentum $p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q}$

Hamiltonian

$$\mathcal{H} = T(\dot{q} = p/m) + V(q) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

$$\left.\begin{aligned} \dot{q} &= \frac{\partial \mathcal{H}}{\partial p} = p/m \\ \dot{p} &= -\frac{\partial \mathcal{H}}{\partial q} = -m\omega^2 q \end{aligned}\right\} \Rightarrow \ddot{q} + \omega^2 q = 0$$

solution

Scaled variables

$Q \equiv q/q_0, \quad P = p/p_0$

$$\alpha = Q + iP \begin{cases} Q = \mathrm{Re}[\alpha] \\ P = \mathrm{Im}[\alpha] \\ \mathcal{H} = E_0 \alpha^* \alpha \end{cases}$$

Phase plane

Quantum Harmonic Oscillator

Formal Quantization Procedure:

$$\boxed{q \to \hat{q}, \quad p \to \hat{p}, \quad [\hat{q}, \hat{p}] = i\hbar}$$

Choose $E_0 = \hbar\omega \Rightarrow q_0 = \sqrt{\frac{2\hbar}{m\omega}}, \quad p_0 = \sqrt{2m\hbar\omega}$

natural scale

$$\alpha \to \hat{a} = \hat{Q} + i\hat{P} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} + i\frac{\hat{p}}{m\omega}\right)$$

$$[\hat{a}, \hat{a}^+] = 1$$

Rewrite:

$$\hat{H} = \hbar\omega(\hat{Q}^2 + \hat{P}^2) = \hbar\omega(\hat{a}^+\hat{a} + 1/2)$$

$\hat{N} = \hat{a}^+\hat{a}$ **(number operator)**

Quantum Electrodynamics – Intro to Field Theory

Quantum Harmonic Oscillator

Formal Quantization Procedure:

$$q \to \hat{q}, \quad p \to \hat{p}, \quad [\hat{q}, \hat{p}] = i\hbar$$

Choose $E_0 = \hbar\omega$ ⇒ $q_0 = \sqrt{\frac{2\hbar}{m\omega}}, \; p_0 = \sqrt{2m\hbar\omega}$

natural scale

$$\alpha \to \hat{a} = \hat{Q} + i\hat{P} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} + i\frac{\hat{p}}{m\omega}\right)$$

$$[\hat{a}, \hat{a}^+] = 1$$

Rewrite:

$$\hat{H} = \hbar\omega(\hat{Q}^2 + \hat{P}^2) = \hbar\omega(\hat{a}^+\hat{a} + 1/2)$$

$\hat{N} = \hat{a}^+\hat{a}$ **(number operator)**

Commutator $[\hat{H}, \hat{N}] = 0$

⇒ **joint energy/number states** $|n\rangle$

$$\boxed{\begin{aligned} \hat{H}|n\rangle &= \hbar\omega(n+1/2)|n\rangle \\ \hat{N}|n\rangle &= n|n\rangle \end{aligned}}$$

Commutators

$$\left.\begin{aligned} [\hat{N}, \hat{a}^+] &= \hat{a}^+ \\ [\hat{N}, \hat{a}] &= -\hat{a} \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} \hat{a}|n\rangle &= \sqrt{n}\,|n-1\rangle \\ \hat{a}^+|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ \hat{a}|0\rangle &= 0 \end{aligned}}$$

Generating excited states

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^+)^n|0\rangle}$$

Quantum Electrodynamics – Intro to Field Theory

Commutator $[\hat{H}, \hat{N}] = 0$

⇒ joint energy/number states $|n\rangle$

$$\boxed{\begin{aligned} \hat{H}|n\rangle &= \hbar\omega(n+1/2)|n\rangle \\ \hat{N}|n\rangle &= n|n\rangle \end{aligned}}$$

Commutators

$$\left.\begin{aligned} [\hat{N}, \hat{a}^+] &= \hat{a}^+ \\ [\hat{N}, \hat{a}] &= -\hat{a} \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} \hat{a}|n\rangle &= \sqrt{n}\,|n-1\rangle \\ \hat{a}^+|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ \hat{a}|0\rangle &= 0 \end{aligned}}$$

Generating excited states

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^+)^n|0\rangle}$$

Expectation values for $\hat{q}$ and $\hat{p}$ in number states

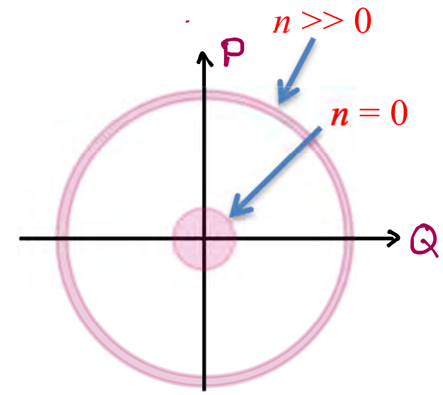
$$\langle n|\hat{q}|n\rangle = \langle n|\hat{p}|n\rangle = 0$$

$$\langle n|\hat{q}^2|n\rangle = \frac{q_0^2}{2}(n+1/2) \neq 0$$

$$\langle n|\hat{p}^2|n\rangle = \frac{p_0^2}{2}(n+1/2) \neq 0$$

$$\Delta q\,\Delta p = \frac{q_0 p_0}{2}(n+1/2) = \hbar(n+1/2)$$

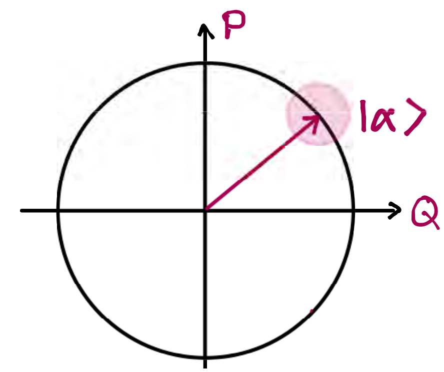
Phase space visualization of number states



Quasi-classical (coherent) state

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_i \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

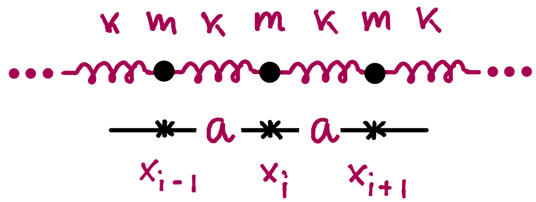
$$\Delta q\,\Delta p = \hbar/2, \quad \Delta Q = \Delta P$$



Quantum Electrodynamics – Intro to Field Theory

Lagrange formulation of 1D Scalar Field

"vibronium"



Configuration space = $\{x_i\}$ **(set of *N* osc. positions)**

$$T=\sum_{i=1}\frac{1}{2}m\dot{x}_i^2, \quad V=\sum_{i=1}^{N}\frac{1}{2}\kappa(x_{i+1}-x_i)^2$$

Lagrangian, equations of motion

Continuum limit ⇨ **Elastic rod**

$N\to\infty$ $\quad m/a \to \mu$ ⟵ linear mass density

$a\to dx$ $\quad \kappa a \to Y$ ⟵ Youngs modulus

$\{x_i\} \to y(x)$ ⟵ displacement field (sound)

Rewrite

$$T=\lim_{N\to\infty}\sum_{i=1}^{N} a\frac{1}{2}\left(\frac{m}{a}\right)\dot{x}_i^2=\int_0^L dx\,\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2$$

$$V=\lim_{N\to\infty}\sum_{i=1}^{N} a\frac{1}{2}\kappa a\left(\frac{x_{i+1}-x_i}{a}\right)^2=\int_0^L dx\,\frac{1}{2}Y\left(\frac{\partial y}{\partial x}\right)^2$$

Lagrangian:

$$\boxed{\mathcal{L}=T-V=\int_0^L dx\,\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2-\int_0^L dx\,\frac{1}{2}Y\left(\frac{\partial y}{\partial x}\right)^2}$$

Notes, Homework ⇨ **Scalar wave equation**

$$\boxed{\frac{\partial^2 y}{\partial t^2}-\frac{Y}{\mu}\frac{\partial y^2}{\partial x^2}=0}$$

– Not yet ready for Quantization –

Quantum Electrodynamics – Intro to Field Theory

Rewrite

$$T = \lim_{N\to\infty} \sum_{i=1}^{N} a\,\frac{1}{2}\left(\frac{m}{a}\right)\dot{x}_i^2 = \int_0^L dx\,\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2$$

$$V = \lim_{N\to\infty} \sum_{i=1}^{N} a\,\frac{1}{2}\kappa a\left(\frac{x_{i+1}-x_i}{a}\right)^2 = \int_0^L dx\,\frac{1}{2}Y\left(\frac{\partial y}{\partial x}\right)^2$$

Lagrangian:

$$\mathcal{L} = T - V = \int_0^L dx\,\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 - \int_0^L dx\,\frac{1}{2}Y\left(\frac{\partial y}{\partial x}\right)^2$$

Notes, Homework ⇨ **Scalar wave equation**

$$\frac{\partial^2 y}{\partial t^2} - \frac{Y}{\mu}\frac{\partial y^2}{\partial x^2} = 0$$

– Not yet ready for Quantization –

Normal Mode Decomposition

Field in cavity:

Solutions to wave eq.

Let $y(x,t) = g(t)u(x) = g_0 e^{i\omega t}u(x)$ ⇨

$\ddot{y} - v^2 y'' = -\omega^2 g(t)u(x) - v^2 g(t)u''(x) = 0$

$$u''(x) = -k^2 u(x), \quad k = \omega/v$$

Solutions in cavity:

$$\boxed{u_k(x) = \sqrt{\frac{2}{L}}\sin(kx), \quad k = \frac{n\pi}{L}}$$

These standing waves are a set of Normal Modes

Quantum Electrodynamics – Intro to Field Theory

Normal Mode Decomposition

Field in cavity:

Solutions to wave eq.

Let $y(x,t) = g(t)u(x) = g_0 e^{i\omega t} u(x)$

$$\ddot{y} - v^2 y'' = -\omega^2 g(t)u(x) - v^2 g(t)u''(x) = 0$$

$$\boxed{u''(x) = -k^2 u(x), \quad k = \omega/v}$$

Solutions in cavity:

$$\boxed{u_k(x) = \sqrt{\frac{2}{L}}\sin(kx), \quad k = \frac{n\pi}{L}}$$

These standing waves are a set of Normal Modes

These modes are orthonormal and complete

$$\boxed{y(x,t) = \sqrt{L}\sum_k q_k(t)\, u_k(x)}$$

Normal mode expansion of $y(x,t)$ in basis $u_k(x)$

Lagrangian for the acoustic field:

$$T = \int_0^L dx\, \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 = \sum_{k,k'} \frac{1}{2}\underbrace{\mu L}_{M}\, \dot{q}_k \dot{q}_{k'} \underbrace{\int_0^L dx\, u_k(x) u_{k'}(x)}_{\delta_{kk'}}$$
$$= \sum_k \frac{1}{2} M \dot{q}_k^2$$

$$V = \int_0^L dx\, \frac{1}{2} Y\left(\frac{\partial y}{\partial x}\right)^2 = \sum_{k,k'} \frac{1}{2} Y L\, q_k q_{k'} \int_0^L dx \left(\frac{\partial u_k}{\partial x}\right)\left(\frac{\partial u_{k'}}{\partial x}\right)$$
$$= \sum_k \frac{1}{2} M \omega_k^2 q_k^2$$

Quantum Electrodynamics – Intro to Field Theory

Normal Mode Decomposition

Field in cavity:

Solutions to wave eq.

Let $y(x,t) = g(t)u(x) = g_0 e^{i\omega t} u(x)$

$$\ddot{y} - v^2 y'' = -\omega^2 g(t) u(x) - v^2 g(t) u''(x) = 0$$

$$\boxed{u''(x) = -k^2 u(x), \quad k = \omega / v}$$

Solutions in cavity:

$$\boxed{u_k(x) = \sqrt{\frac{2}{L}} \sin(kx), \quad k = \frac{n\pi}{L}}$$

These standing waves are a set of Normal Modes

These modes are orthonormal and complete

$$\boxed{y(x,t) = \sqrt{L} \sum_k q_k(t) u_k(x)}$$

Normal mode expansion of $y(x,t)$ in basis $u_k(x)$

Lagrangian for the acoustic field:

$$T = \int_0^L dx \frac{1}{2} \mu \left(\frac{\partial y}{\partial t}\right)^2 = \sum_{k,k'} \frac{1}{2} \underbrace{\mu L}_{M} \dot{q}_k \dot{q}_{k'} \underbrace{\int_0^L dx\, u_k(x) u_{k'}(x)}_{\delta_{kk'}}$$

$$= \sum_k \frac{1}{2} M \dot{q}_k^2$$

$$V = \int_0^L dx \frac{1}{2} Y \left(\frac{\partial y}{\partial x}\right)^2 = \sum_{k,k'} \frac{1}{2} Y L q_k q_{k'} \int_0^L dx \left(\frac{\partial u_k}{\partial x}\right)\left(\frac{\partial u_{k'}}{\partial x}\right)$$

$$= \sum_k \frac{1}{2} M \omega_k^2 q_k^2$$

$$\boxed{\mathcal{L} = T - V = \sum_k \left(\frac{1}{2} M \dot{q}_k^2 - \frac{1}{2} M \omega_k^2 q_k^2\right) = \sum_k \mathcal{L}_k}$$

Quantum Electrodynamics – Intro to Field Theory

The rest now follows from the Lagrangian

$$\mathcal{L} = T - V = \sum_k \left(\tfrac{1}{2} M \dot{q}_k^2 - \tfrac{1}{2} M \omega_k^2 q_k^2 \right) = \sum_k \mathcal{L}_k$$

Canonical Momentum

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = M \dot{q}_k$$

Hamiltonian

$$\mathcal{H}(\{p_k, q_k\}) = T + V = \sum_k \left(\frac{p_k^2}{2M} + \tfrac{1}{2} M \omega_k^2 q_k^2 \right)$$

(collection of SHO's, one for each normal mode)

Following the standard recipe…

$$E_{0,k} = \hbar\omega_k, \quad q_{0,k} = \sqrt{2\hbar/M\omega_k}, \quad p_{0,k} = \sqrt{2M\hbar\omega_k}$$

$$Q_k = q_k/q_{0,k}, \quad P_k = p_k/p_{0,k}, \quad \alpha_k = Q_k + iP_k$$

… we get solutions

$$\alpha_k(t) = Q_k(t) + i P_k(t) = \alpha_k(0)\, e^{-i\omega_k t}$$

This finally gives us

$$\mathcal{H} = \sum_k \hbar\omega_k \left(Q_k^2 + P_k^2 \right) = \sum_k \hbar\omega_k \alpha_k^* \alpha_k$$

$$y(x,t) = \sqrt{L} \sum_k q_k(t)\, u_k(x)$$

$$= \frac{1}{2} \sum_k \sqrt{L q_{0,k}^2} \left(\alpha_k(t)\, u_k(x) + \alpha_k^*(t)\, u_k(x) \right)$$

$$\alpha = Q + iP \qquad Q = \tfrac{1}{2}(\alpha + \alpha^*) \qquad q = \tfrac{q_0}{2}(\alpha + \alpha^*)$$

$$\alpha^* = Q - iP \qquad P = \tfrac{1}{2i}(\alpha - \alpha^*) \qquad p = \tfrac{p_0}{2}(\alpha - \alpha^*)$$

Quantum Electrodynamics – Intro to Field Theory

... we get solutions

$$\alpha_k(t) = Q_k(t) + i P_k(t) = \alpha_k(0)\, e^{-i\omega_k t}$$

This finally gives us

$$\mathcal{H} = \sum_k \hbar\omega_k \left(Q_k^2 + P_k^2\right) = \sum_k \hbar\omega_k\, \alpha_k^* \alpha_k$$

$$y(x,t) = \sqrt{L} \sum_k q_k(t)\, u_k(x)$$

$$= \frac{1}{2} \sum_k \sqrt{L q_{0,k}^2}\left(\alpha_k(t)\, u_k(x) + \alpha_k^*(t)\, u_k(x)\right)$$

Formal Quantization Procedure:

$$q_k \to \hat{q}_k \;, \quad p_k \to \hat{p}_k \;, \quad \alpha_k \to \hat{a}_k$$

$$[\hat{q}_k, \hat{p}_{k'}] = i\hbar\, \delta_{kk'} \,, \quad [\hat{a}_k, \hat{a}^\dagger_{k'}] = \delta_{kk'} \,, \quad [\hat{a}_k, \hat{a}_{k'}] = 0$$

Note: $k \neq k'$ ➡ operators commute

(normal modes = independent degs. of freedom)

Hamiltonian & Quantized fields

$$\hat{H} = \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + 1/2\right)$$

$$\hat{y}(x) = \sqrt{L} \sum_k \hat{q}_k\, u_k(x) = \sum_k \sqrt{L q_{0,k}^2}\left(\hat{a}_k u_k(x) + \hat{a}_k^\dagger u_k(x)\right)$$

$$\hat{\Pi}(x) = \frac{1}{\sqrt{L}} \sum_k \hat{p}_k\, u_k(x) = -i \sum_k \sqrt{\frac{p_{0,k}^2}{L}}\left(\hat{a}_k u_k(x) - \hat{a}_k^\dagger u_k(x)\right)$$

field $\hat{y}(x)$ and canonical momentum field $\hat{\Pi}(x)$

➡ $[\hat{y}(x), \hat{\Pi}(x')] = i\hbar\, \delta(x - x')$

Quantum Electrodynamics – Intro to Field Theory

Formal Quantization Procedure:

$$q_k \to \hat{q}_k \ , \quad p_k \to \hat{p}_k \ , \quad a_k \to \hat{a}_k$$

$$[\hat{q}_k, \hat{p}_{k'}] = i\hbar\,\delta_{kk'} \ , \quad [\hat{a}_k, \hat{a}^+_{k'}] = \delta_{kk'} \ , \quad [\hat{a}_k, \hat{a}_{k'}] = 0$$

Note: $k \neq k'$ ⇨ operators commute

(normal modes = independent degs. of freedom)

Hamiltonian & Quantized fields

$$\hat{H} = \sum_k \hbar\omega_k \left(\hat{a}^+_k \hat{a}_k + 1/2\right)$$

$$\hat{y}(x) = \sqrt{L}\sum_k \hat{q}_k\, u_k(x) = \sum_k \sqrt{L q_{0,k}^2}\left(\hat{a}_k u_k(x) + \hat{a}^+_k u_k(x)\right)$$

$$\hat{\Pi}(x) = \frac{1}{\sqrt{L}}\sum_k \hat{p}_k\, u_k(x) = -i\sum_k \sqrt{\frac{p_{0,k}^2}{L}}\left(\hat{a}_k u_k(x) - \hat{a}^+_k u_k(x)\right)$$

field $\hat{y}(x)$ and canonical momentum field $\hat{\Pi}(x)$

⇨ $[\hat{y}(x), \hat{\Pi}(x')] = i\hbar\,\delta(x-x')$

Quantum States:

The quantum field is a collection of QSHO's

⇩

State space = tensor product of SHO spaces

Fock Space $\quad \mathcal{E} = \mathcal{E}_{k_1} \otimes \mathcal{E}_{k_2} \otimes \mathcal{E}_{k_3} \otimes \cdots , \mathcal{E}_{k_j}$ ← SHO space

Fock State $\quad |\{n_{k_1}, n_{k_2}, \ldots\}\rangle = |n_{k_1}\rangle \otimes |n_{k_2}\rangle \otimes \ldots |n_{k_j}\rangle$

$\hat{a}_{k_i}, \ \hat{a}^+_{k_i}$ destroy/create excitations in mode k_i

Vacuum State $\quad |0\rangle$ ← zero quanta in every mode

Favorite Question: **What is a Phonon?**

Quantum Electrodynamics – Intro to Field Theory

Quantum States:

The quantum field is a collection of QSHO's

State space = tensor product of SHO spaces

Fock Space $\mathcal{E} = \mathcal{E}_{k_1} \otimes \mathcal{E}_{k_2} \otimes \mathcal{E}_{k_3} \otimes \dots, \mathcal{E}_{k_j}$ ← SHO space

Fock State $|\{n_{k_1}, n_{k_2}, \dots\}\rangle = |n_{k_1}\rangle \otimes |n_{k_2}\rangle \otimes \dots |n_{k_j}\rangle$ ← SHO state

$\hat{a}_{k_i}, \hat{a}_{k_i}^{\dagger}$ destroy/create excitations in mode k_i

Vacuum State $|0\rangle$ ← zero quanta in every mode

Favorite Question: **What is a Phonon?**

Vacuum Fluctuations:

Expectation value of the Field

$$\langle 0|\hat{\eta}(x)|0\rangle =$$

$$\sum_k \frac{1}{2}\sqrt{L q_{0,k}^2}\left(\langle 0|\hat{a}_k|0\rangle u_k(x) + \langle 0|\hat{a}_k^{\dagger}|0\rangle u_k^*(x)\right) = 0$$

Zero Point Fluctuations

$$\langle 0|\hat{\eta}(x)^2|0\rangle$$

$$= \sum_k \frac{1}{4} L q_{0,k} q_{0,k} \langle 0|\hat{a}_k^{\dagger}\hat{a}_k + 1|0\rangle u_k(x) u_k^*(x)$$

$$= \sum_k \frac{1}{4} L q_{0,k}^2 |u_k(x)|^2 \neq 0$$

Thus $\Delta\eta(x) \neq 0$ with zero phonons in field

Note the famous divergence:

$$E_{vac} = \langle 0|\hat{H}|0\rangle = \sum_k \frac{\hbar\omega_k}{2} \rightarrow \infty \text{ for } k \rightarrow \infty$$

Quantum Electrodynamics – Intro to Field Theory

Vacuum Fluctuations:

Expectation value of the Field

$$\langle 0|\hat{\eta}(x)|0\rangle = \sum_k \frac{1}{2}\sqrt{L q_{0,k}^2}\left(\langle 0|\hat{a}_k|0\rangle u_k(x) + \langle 0|\hat{a}_k^+|0\rangle u_k(x)\right) = 0$$

Zero Point Fluctuations

$$\langle 0|\hat{\eta}(x)^2|0\rangle$$

$$= \sum_k \frac{1}{4} L q_{0,k} q_{0,k} \langle 0|\hat{a}_k^+\hat{a}_k + 1|0\rangle u_k(x) u_k(x)$$

$$= \sum_k \frac{1}{4} L q_{0,k}^2 |u_k(x)|^2 \neq 0$$

Thus $\Delta\eta(x) \neq 0$ **with zero phonons in field**

Note the famous divergence:

$$E_{vac} = \langle 0|\hat{H}|0\rangle = \sum_k \frac{\hbar\omega_k}{2} \to \infty \text{ for } k \to \infty$$

Are our Phonons waves or particles?

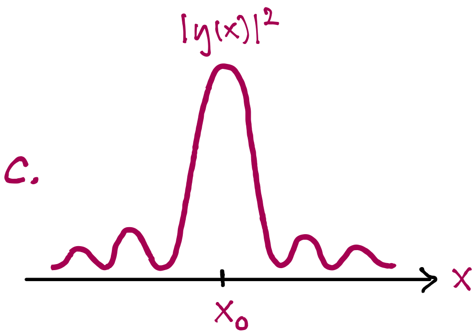
Extended

Localized

Particle-like Phonons

Classical Wavepacket

$$\eta(x) = \sum_k f_k u_k(x) + c.c.$$



Define $\hat{A}^+ = \sum_k f_k \hat{a}_k^+$, $\sum_k |f_k|^2 = 1$

$$\Rightarrow \hat{A}^+|0\rangle = f_{k_1}|1_{k_1}, 0_{k_2}, \ldots\rangle + f_{k_2}|0_{k_1}, 1_{k_2}, \ldots\rangle + \ldots$$

Localized excitation in the field.

These Particle-like Phonons are Bosons

$$\hat{A}'^+\hat{A}^+|0\rangle = \hat{A}^+\hat{A}'^+|0\rangle$$

1st particle @ *x*
2nd particle @ *x'*

1st particle @ *x'*
2nd particle @ *x*

Quantum Electrodynamics – Intro to Field Theory

What is Quantum Optics?

– Something both different and more than classical optics
– The science of non-classical light
– Any science that combines light and quantum mechanics

What is Light?

– Electromagnetic waves?
– Photons (particles)?

Let's take a quick poll…

Not so fast!

Anti-photon

W.E. Lamb, Jr. **(Nobel Prize in Physics 1955)**

Optical Sciences Center, University of Arizona, Tucson, AZ 85721, USA

Received: 23 July 1994 / Accepted: 18 September 1994

Abstract. It should be apparent from the title of this article that the author does not like the use of the word "photon", which dates from 1926. In his view, there is no such thing as a photon. Only a comedy of errors and historical accidents led to its popularity among physicists and optical scientists. I admit that the word is short and convenient. Its use is also habit forming. Similarly, one might find it convenient to speak of the "aether" or "vacuum" to stand for empty space, even if no such thing existed. There are very good substitute words for "photon", (e.g., "radiation" or "light"), and for "photonics" (e.g., "optics" or "quantum optics"). Similar objections are possible to use of the word "phonon", which dates from 1932. Objects like electrons, neutrinos of finite rest mass, or helium atoms can, under suitable conditions, be considered to be particles, since their theories then have viable non-relativistic and non-quantum limits. This paper outlines the main features of the quantum theory of radiation and indicates how they can be used to treat problems in quantum optics.

afterward, there was a population explosion of people engaged in fundamental research and in very useful technical and commercial developments of lasers. QTR was available, but not in a form convenient for the problems at hand. The photon concepts as used by a high percentage of the laser community have no scientific justification. It is now about thirty-five years after the making of the first laser. The sooner an appropriate reformulation of our educational processes can be made, the better.

1 A short history of pre-photonic radiation

Modern optical theory [2] began with the works of Ch. Huyghens and I. Newton near the end of the seventeenth century. Huyghen's treatise on wave optics was published in 1690. Newton's "Optiks", which appeared in 1704, dealt with his corpuscular theory of light.

A decisive work in 1801 by T. Young, on the two-slit diffraction pattern, showed that the wave version of optics

With all due respect to Prof. Lamb…

What is light?

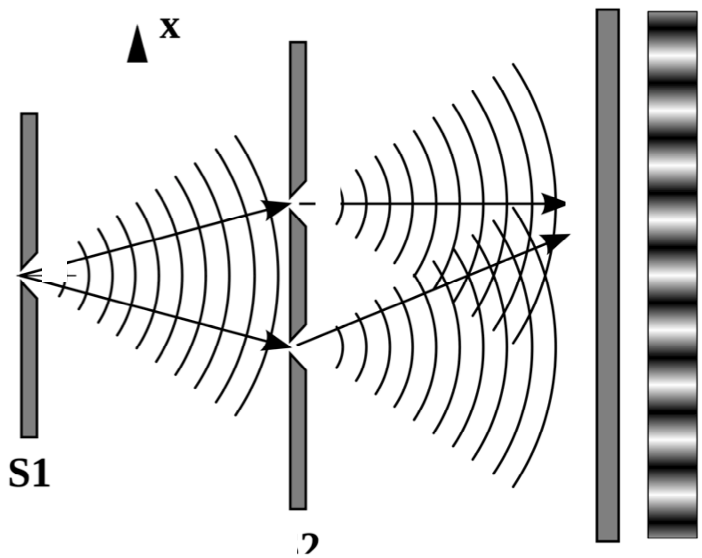
– a wave?
– a stream of particles (photons)?

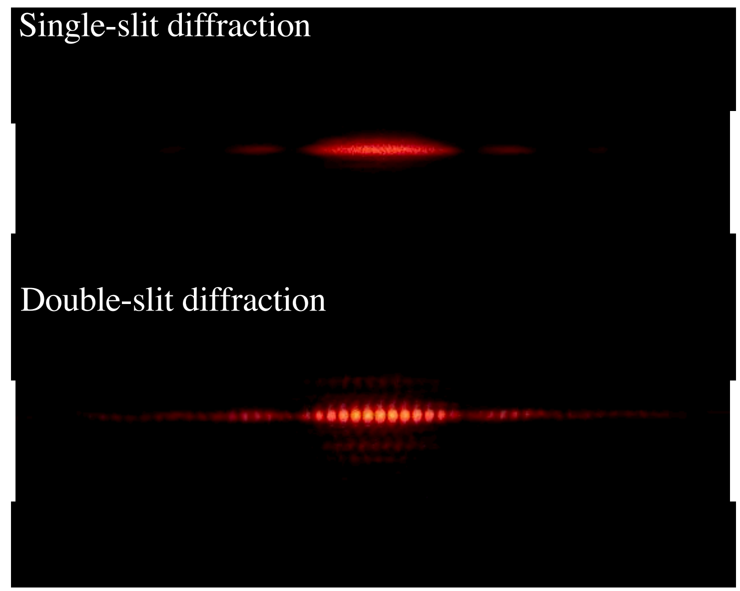
Take the question seriously

– test each hypothesis through experimentation!

Key signature of wave behavior? – Interference!

Double-slit experiment



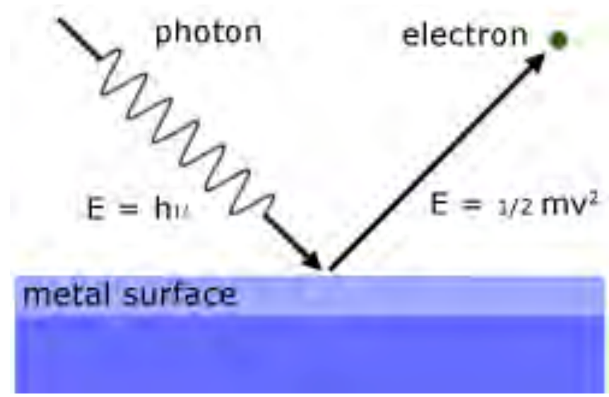


Key signature of particle behavior?

Einstein: Photo-Electric Effect

Electrons are released only for light with a frequency ν such that $h\nu$ is greater than the *work function* of the metal in question

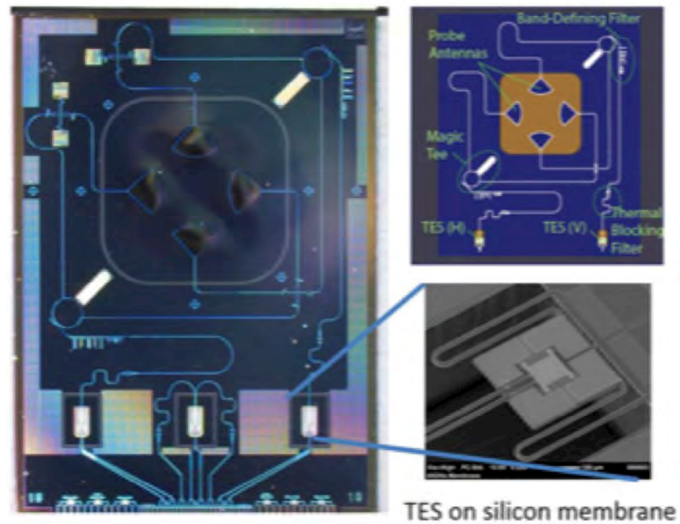
The quantum theory of electron excitation can explain this based on classical electromagnetic fields, so the photo-electric effect only confirms that *charge* is quantized.



Key signature of particle behavior?

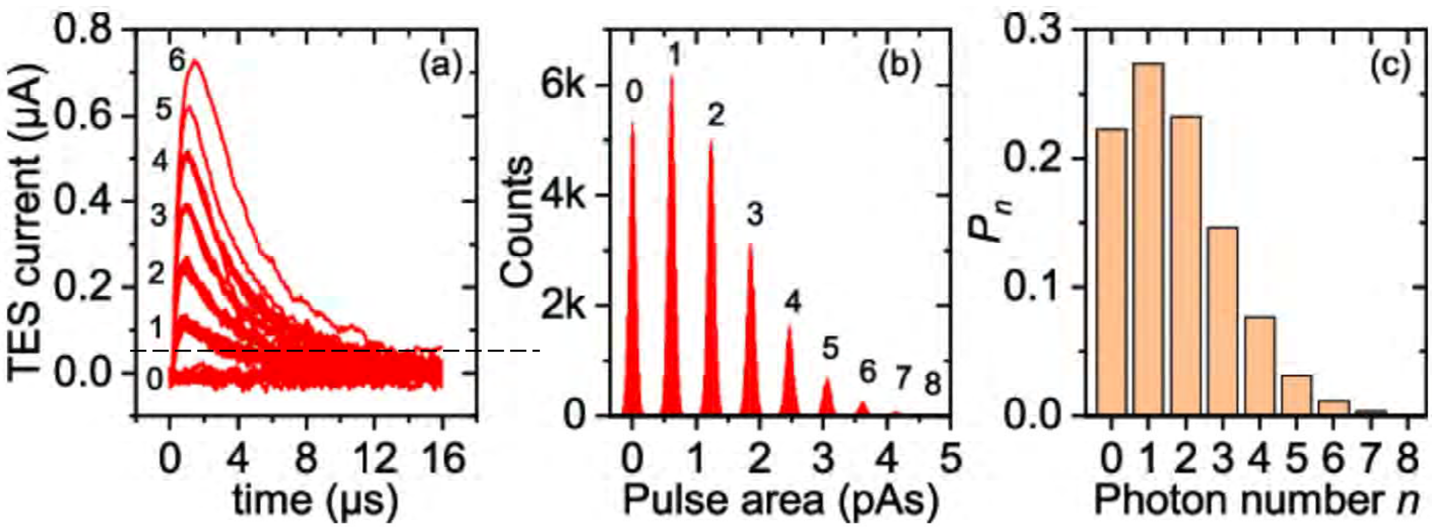
Transition Edge Sensors

Superconducting calorimeter



TES Output

Attenuated laser pulses

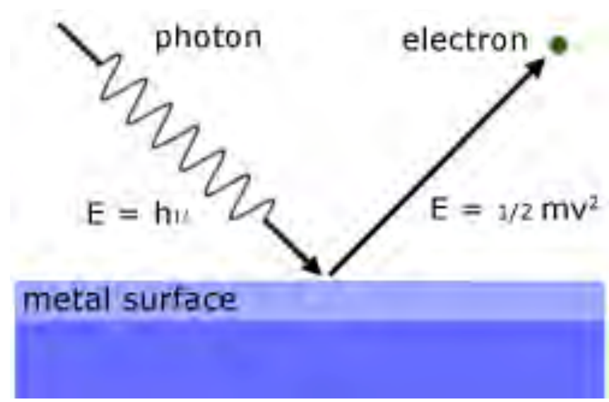


Key signature of particle behavior?

Einstein: Photo-Electric Effect

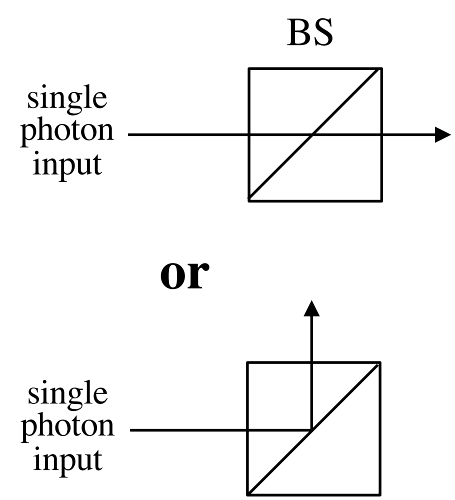
Electrons are released only for light with a frequency ν such that $h\nu$ is greater than the *work function* of the metal in question

The quantum theory of electron excitation can explain this based on classical electromagnetic fields, so the photo-electric effect only confirms that *charge* is quantized.

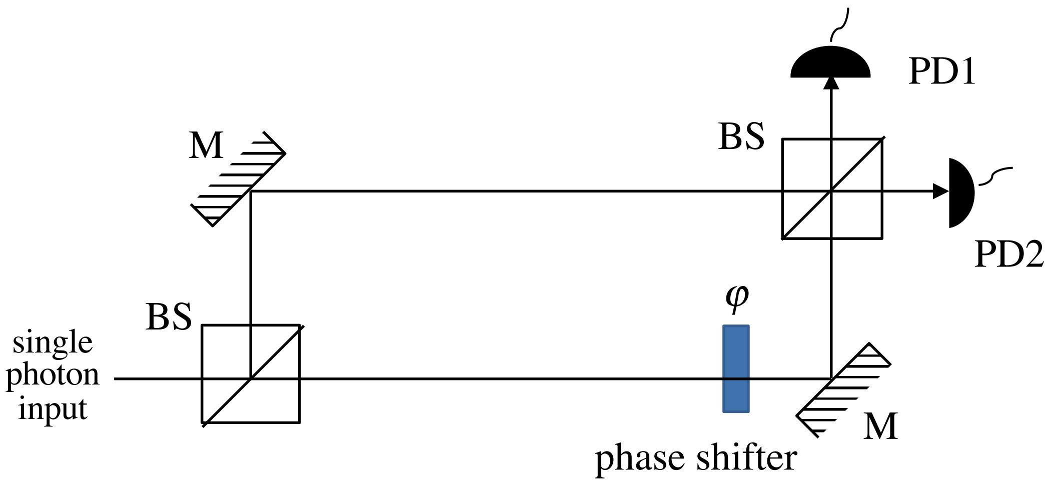


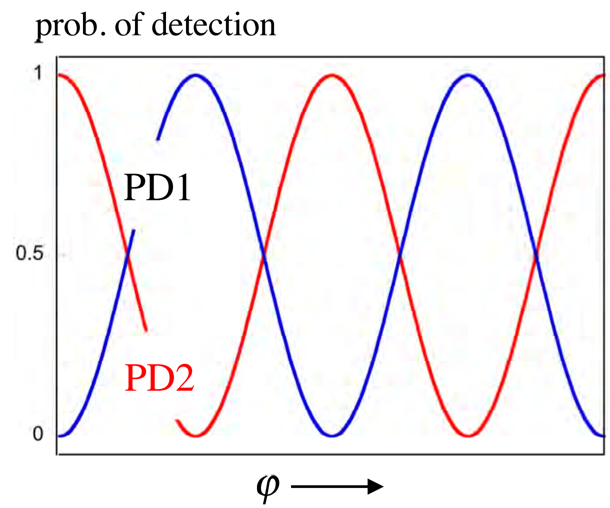
Indivisibility!

A particle incident on a barrier is either transmitted or reflected.

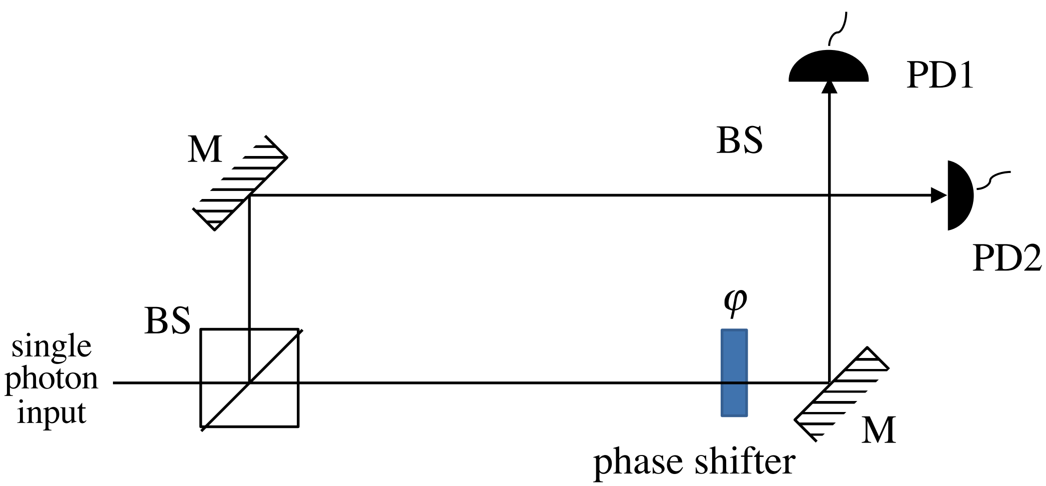


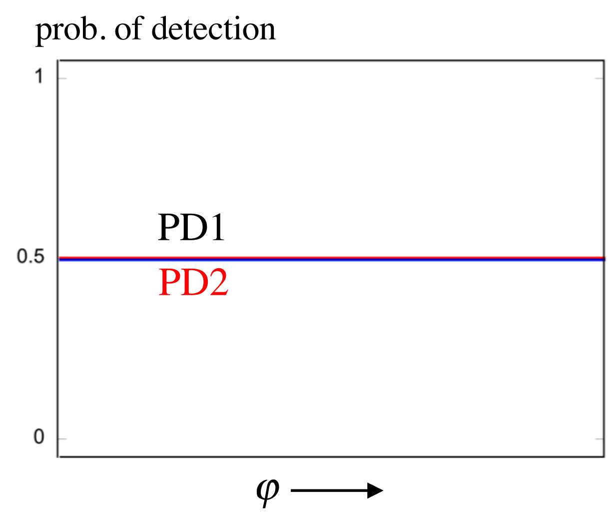
Wave behavior





Particle behavior





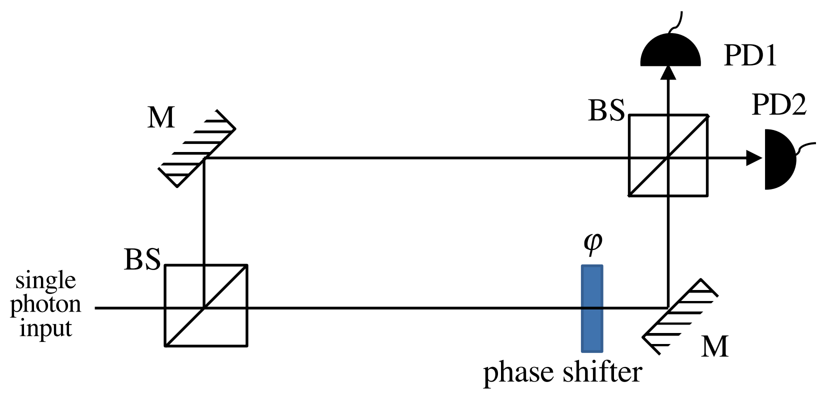
– Evidently a single photon can behave like a wave or a particle, depending on the experiment we do. This is what we know as *wave-particle duality*.

– Does the photon "know" when it hits the first BS if we are doing a wave or particle experiment and then behaves accordingly?

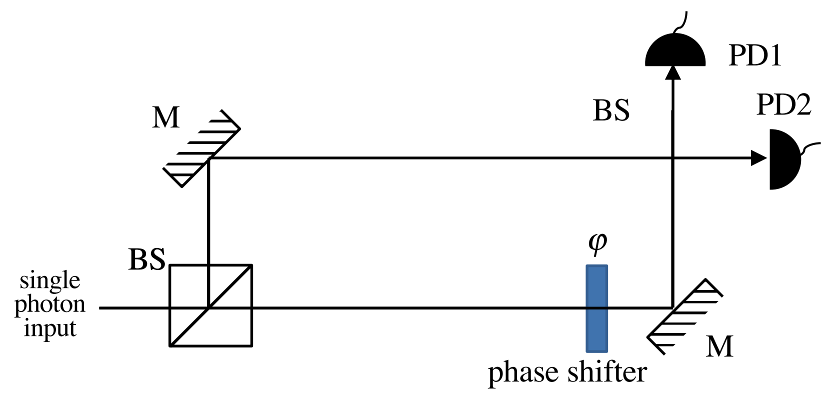
– Wheelers's thought experiment: Delayed Choice!

Decide *at random* whether to put in the second BS only *after* the photon has passed the first BS

Wave behavior



Particle behavior



Wheelers experiment was done in 2008

PRL **100,** 220402 (2008) PHYSICAL REVIEW LETTERS week ending 6 JUNE 2008

Delayed-Choice Test of Quantum Complementarity with Interfering Single Photons

Vincent Jacques,[1] E Wu,[1,2] Frédéric Grosshans,[1] François Treussart,[1] Philippe Grangier,[3]
Alain Aspect,[3] and Jean-François Roch[1,*]

[1]*Laboratoire de Photonique Quantique et Moléculaire, Ecole Normale Supérieure de Cachan, UMR CNRS 8537, Cachan, France*
[2]*State Key Laboratory of Precision Spectroscopy, East China Normal University, Shanghai, China*
[3]*Laboratoire Charles Fabry de l'Institut d'Optique, UMR CNRS 8501, Palaiseau, France*
(Received 12 February 2008; published 3 June 2008)

We report an experimental test of quantum complementarity with single-photon pulses sent into a Mach-Zehnder interferometer with an output beam splitter of adjustable reflection coefficient R. In addition, the experiment is realized in Wheeler's delayed-choice regime. Each randomly set value of R allows us to observe interference with visibility V and to obtain incomplete which-path information characterized by the distinguishability parameter D. Measured values of V and D are found to fulfill the complementarity relation $V^2 + D^2 \leq 1$.

Wheelers experiment was done in 2008

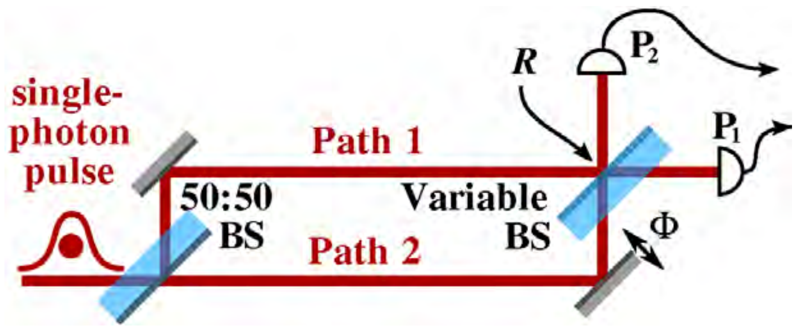


FIG. 1 (color online). Delayed-choice complementarity-test experiment. A single-photon pulse is sent into a Mach-Zehnder interferometer, composed of a 50/50 input beam splitter (BS) and a variable output beam splitter (VBS). The reflection coefficient is randomly set either to the null value or to an adjustable value R, after the photon has entered the interferometer. The single-photon photodetectors P_1 and P_2 allow to record both the interference and the WPI.

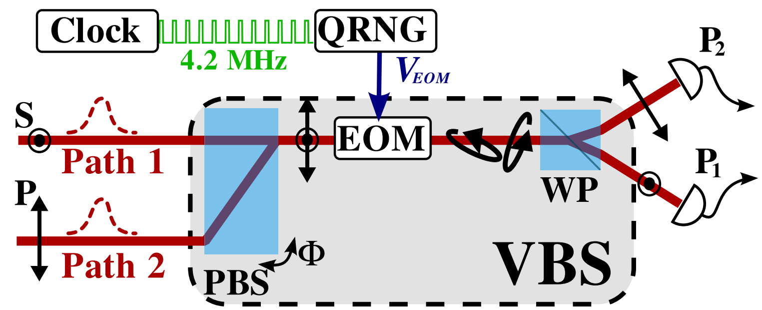


FIG. 2 (color online). Variable output beam splitter (VBS) implementation. The optical axis of the polarization beam splitter (PBS) and the polarization eigenstates of the Wollaston prism (WP) are aligned, and make an angle β with the optical axis of the EOM. The voltage V_{EOM} applied to the EOM is randomly chosen accordingly to the output of a Quantum Random Number Generator (QRNG), located at the output of the interferometer and synchronized on the 4.2-MHz clock that triggers the single-photon emission.

Wheelers experiment was done in 2008

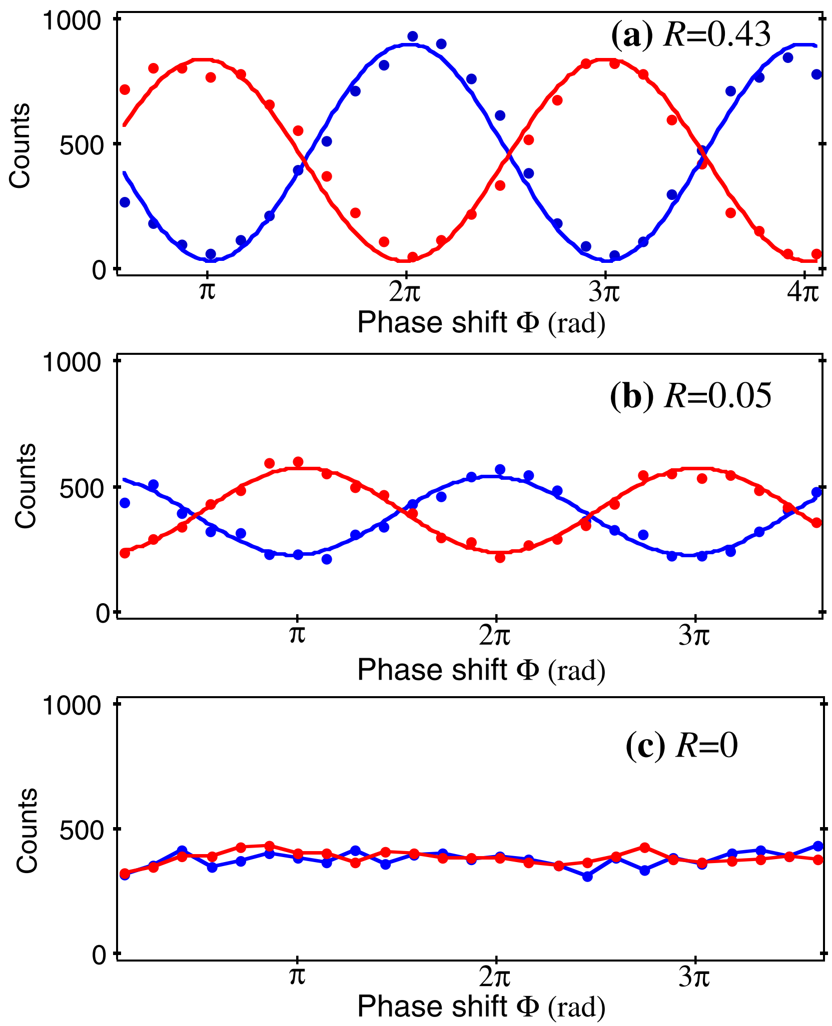


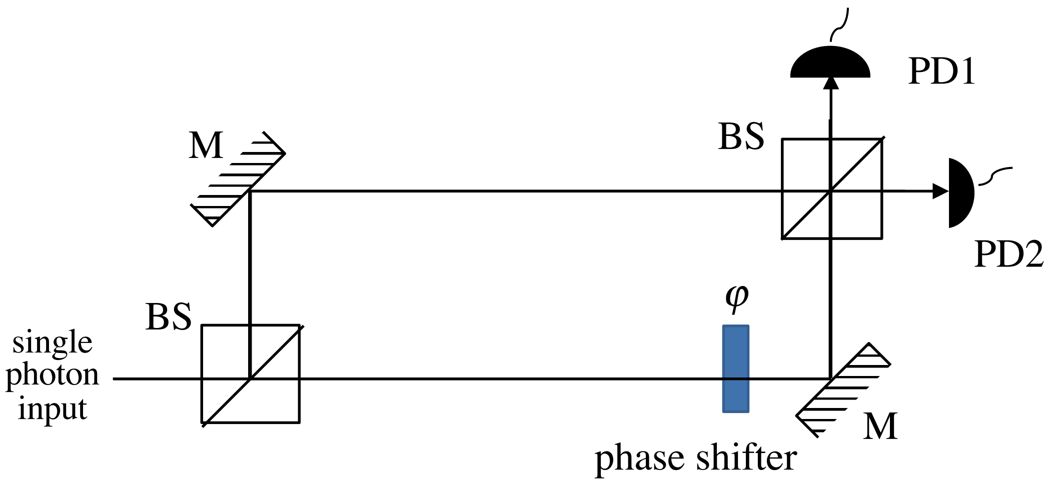
FIG. 3 (color online). Interference visibility V measured in the delayed-choice regime for different values of V_{EOM}. (a)–(c) correspond to $V_{\mathrm{EOM}} \approx 150$ V ($R = 0.43$ and $V = 93 \pm 2\%$), $V_{\mathrm{EOM}} \approx 40$ V ($R = 0.05$ and $V = 42 \pm 2\%$), and $V_{\mathrm{EOM}} = 0$ ($R = 0$ and $V = 0$). Each point is recorded with 1.9 s acquisition time. Detectors dark counts, corresponding to a rate of 60 s^{-1} for each, have been substracted to the data.

Light is *both* a particle and a wave *at the same time*.

What *property we see* depends on the *property we measure*.

This is totally in line with our general quantum theory.

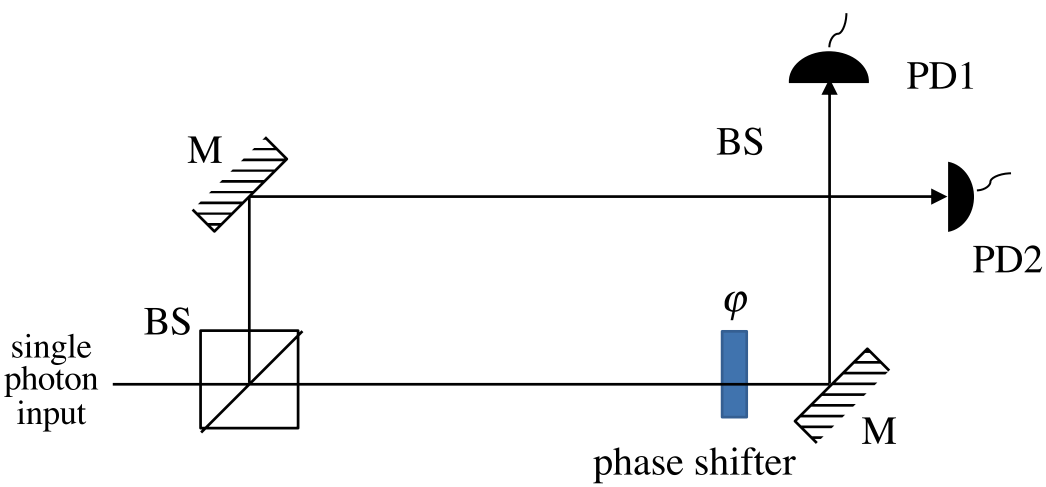
Wave behavior



Both behave the same way with

- **a laser pulse (coherent state)**
- **a pulse of classical light…**

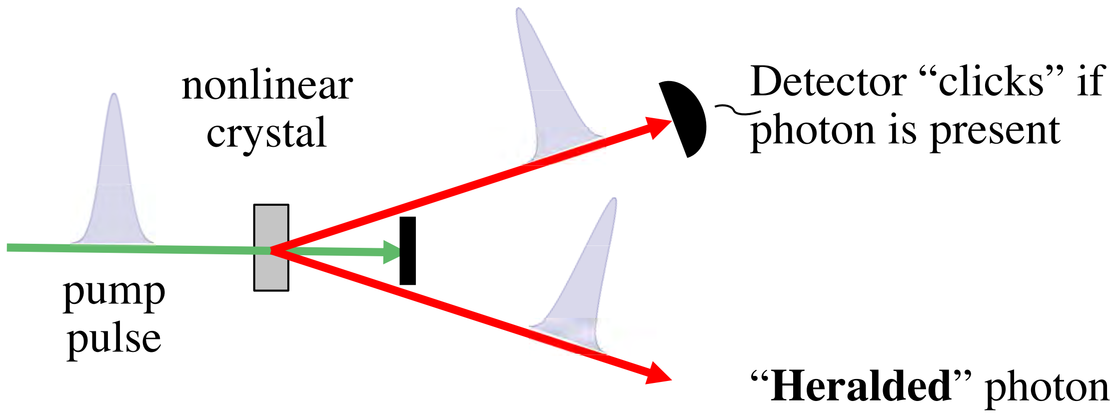
Particle behavior



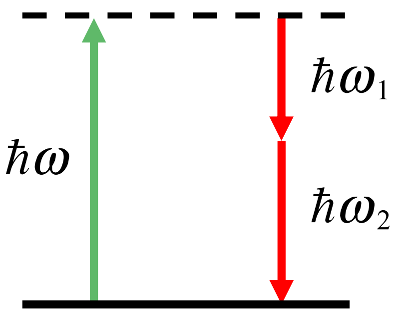
A single photon input is essential….

Making Single Photons

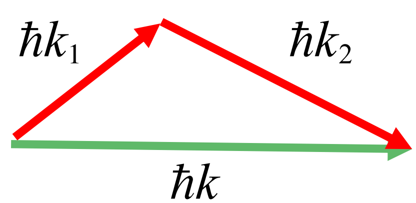
Spontaneous Parametric Down Conversion



Energy conservation: $\hbar\omega = \hbar\omega_1 + \hbar\omega_2$



Momentum conservation: $\hbar k = \hbar k_1 + \hbar k_2$



Quantum Electrodynamics – Intro to Field Theory

Quantum Electrodynamics – QED

Starting point: Maxwells Equations

(1) $\nabla \cdot \vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0}\rho(\vec{r},t)$

(2) $\nabla \cdot \vec{B}(\vec{r},t) = 0$

(3) $\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t)$

(4) $\nabla \times \vec{B}(\vec{r},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{r},t) + \frac{1}{\varepsilon_0 c^2}\vec{j}(\vec{r},t)$

Implicit: Charges & Fields in Vacuum
No “medium response”

Same issue as with our introductory example:

Maxwells eqs are non-local

We need to put the classical description in proper form -> Normal Mode expansion

Free Fields - Switch to Fourier Domain

(1) $i\vec{k} \cdot \vec{E}(\vec{k},t) = \frac{1}{\varepsilon_0}\rho(\vec{k},t)$

(2) $i\vec{k} \cdot \vec{B}(\vec{k},t) = 0$

(3) $i\vec{k} \times \vec{E}(\vec{k},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{k},t)$

(4) $i\vec{k} \times \vec{B}(\vec{k},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{k},t) + \frac{1}{\varepsilon_0 c^2}\vec{j}(\vec{k},t)$

Fourier Transform:
$$\begin{cases} \nabla \cdot \vec{G} \rightleftarrows i\vec{k} \cdot \vec{G} \\ \nabla \times \vec{G} \rightleftarrows i\vec{k} \times \vec{G} \end{cases}$$

Note: This is a Normal Mode decomposition

No charges -> No coupling between modes with different $\vec{k}$

Quantum Electrodynamics – QED

Free Fields - Switch to Fourier Domain

(1) $i\vec{k}\cdot\vec{\mathcal{E}}(\vec{k},t) = \frac{1}{\varepsilon_0}\rho(\vec{k},t)$

(2) $i\vec{k}\cdot\vec{B}(\vec{k},t) = 0$

(3) $i\vec{k}\times\vec{\mathcal{E}}(\vec{k},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{k},t)$

(4) $i\vec{k}\times\vec{B}(\vec{k},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathcal{E}}(\vec{k},t) + \frac{1}{\varepsilon_0 c^2}\vec{j}(\vec{k},t)$

Fourier Transform:
$$\begin{cases} \nabla\cdot\vec{G} \rightleftarrows i\vec{k}\cdot\vec{g} \\ \nabla\times\vec{G} \rightleftarrows i\vec{k}\times\vec{g} \end{cases}$$

<u>Note</u>: This is a Normal Mode decomposition

No charges -> No coupling between modes with different $\vec{k}$

Separate into Transverse & Longitudinal Fields

$$\vec{\mathcal{E}}(\vec{k},t) = \vec{\mathcal{E}}_\parallel(\vec{k},t) + \vec{\mathcal{E}}_\perp(\vec{k},t)$$

$$\vec{B}(\vec{k},t) = \cancel{\vec{B}_\parallel(\vec{k},t)} + \vec{B}_\perp(\vec{k},t) \quad \text{MEq (2)}$$

Entirely Transverse

<u>Note</u>:
- $\vec{\mathcal{E}}_\parallel$ is $\frac{\vec{k}}{k}\times$ the projection of $\vec{\mathcal{E}}$ onto $\vec{k}$
- $\mathcal{E}_\parallel = -\frac{i}{k}\, i\vec{k}\cdot\vec{\mathcal{E}}$ is the projection of $\vec{\mathcal{E}}$ onto $\vec{k}$

$$\vec{\mathcal{E}}_\parallel = \frac{\vec{k}}{k}\vec{\mathcal{E}}_\parallel = \frac{\vec{k}}{k}\left(-\frac{i}{k}\, i\vec{k}\cdot\vec{\mathcal{E}}\right) = \frac{\vec{k}}{\varepsilon_0 k^2}\rho(\vec{k},t)$$

Coulomb field from the charges

Only $\vec{\mathcal{E}}_\perp$ and $\vec{B}_\perp$ are new degrees of freedom beyond the particles -> <u>Free Fields</u>

Quantum Electrodynamics – QED

Separate into Transverse & Longitudinal Fields

$$\vec{\mathcal{E}}(\vec{k},t) = \vec{\mathcal{E}}_{\parallel}(\vec{k},t) + \vec{\mathcal{E}}_{\perp}(\vec{k},t)$$

$$\vec{\mathcal{B}}(\vec{k},t) = \cancel{\vec{\mathcal{B}}_{\parallel}(\vec{k},t)} + \vec{\mathcal{B}}_{\perp}(\vec{k},t)$$ **MEq (2)**

Entirely Transverse

Note:

- $\vec{\mathcal{E}}_{\parallel}$ is $\frac{\vec{k}}{k} \times$ the projection of $\vec{\mathcal{E}}$ onto $\vec{k}$
- $\mathcal{E}_{\parallel} = -\frac{i}{k} i\vec{k}\cdot\vec{\mathcal{E}}$ is the projection of $\vec{\mathcal{E}}$ onto $\vec{k}$

MEq (1)

$$\vec{\mathcal{E}}_{\parallel} = \frac{\vec{k}}{k}\mathcal{E}_{\parallel} = \frac{\vec{k}}{k}\left(-\frac{i}{k}i\vec{k}\cdot\vec{\mathcal{E}}\right) = \frac{\vec{k}}{\varepsilon_0 k^2}\rho(\vec{k},t)$$

Coulomb field from the charges

Only $\vec{\mathcal{E}}_{\perp}$ and $\vec{\mathcal{B}}_{\perp}$ are new degrees of freedom beyond the particles -> <u>Free Fields</u>

Eqs for Transverse Fields, from MEqs (3) & (4)

(5a) $$\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t) = -i\vec{k}\times\vec{\mathcal{E}}_{\perp}(\vec{k},t)$$

(6a) $$\frac{\partial}{\partial t}\vec{\mathcal{E}}_{\perp}(\vec{k},t) = c^2 i\vec{k}\times\vec{\mathcal{B}}(\vec{k},t) - \frac{1}{\varepsilon_0}\vec{j}_{\perp}(\vec{k},t)$$

inverse FT

(5b) $$\frac{\partial}{\partial t}\vec{B}(\vec{r},t) = -\nabla\times\vec{E}_{\perp}(\vec{r},t)$$

(6b) $$\frac{\partial}{\partial t}\vec{E}_{\perp}(\vec{r},t) = c^2\nabla\times\vec{B}(\vec{r},t) - \frac{1}{\varepsilon_0}\vec{j}_{\perp}(\vec{r},t)$$

combine (5b) & (6b)

Wave Equation for the Free Fields

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_{\perp}(\vec{r},t) = \frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t}\vec{j}_{\perp}(\vec{r},t)$$

Quantum Electrodynamics – QED

Eqs for Transverse Fields, from MEqs (3) & (4)

(5a) $\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t) = -i\vec{k}\times\vec{\mathcal{E}}_\perp(\vec{k},t)$

(6a) $\frac{\partial}{\partial t}\vec{\mathcal{E}}_\perp(\vec{k},t) = c^2 i\vec{k}\times\vec{\mathcal{B}}(\vec{k},t) - \frac{1}{\varepsilon_0}\vec{j}_\perp(\vec{k},t)$

inverse FT

(5b) $\frac{\partial}{\partial t}\vec{B}(\vec{r},t) = -\nabla\times\vec{E}_\perp(\vec{r},t)$

(6b) $\frac{\partial}{\partial t}\vec{E}_\perp(\vec{r},t) = c^2\nabla\times\vec{B}(\vec{r},t) - \frac{1}{\varepsilon_0}\vec{j}_\perp(\vec{r},t)$

combine (5b) & (6b)

Wave Equation for the Free Fields

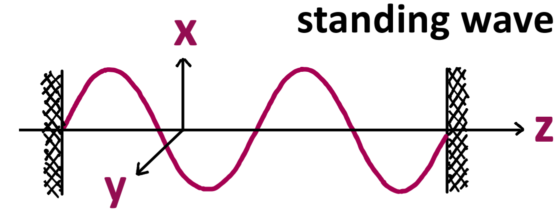
$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_\perp(\vec{r},t) = \frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t}\vec{j}_\perp(\vec{r},t)$$

Normal Modes in a 1D Cavity

Length L

Cross section A

Volume $V = LA$



Normal Modes are Standing Waves

Let $\vec{E}(z,t) = \vec{\varepsilon}_x E_x(z,t)$ and expand

fiducial mass

(7) $E_x(z,t) = \sum_j A_j q_j(t)\sin(k_j z)$, $A_j = \sqrt{\frac{2\omega_j m_j}{\varepsilon_0 V}}$

MEq (4) w/no charges

$$\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}_\perp(\vec{r},t) = \vec{\varepsilon}_x\frac{1}{c^2}\sum_j A_j\dot{q}_j(t)\sin(k_j t)$$

$$= \vec{\varepsilon}_x\left(\cancel{\frac{\partial B_z}{\partial y}} - \frac{\partial B_y}{\partial z}\right)$$

$\mathcal{B}$ transverse ⇒ $B_z = 0$

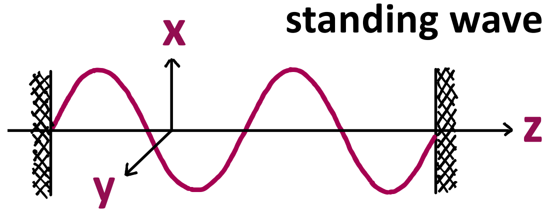
Quantum Electrodynamics – QED

Normal Modes in a 1D Cavity

Length L

Cross section A

Volume $V = LA$



Normal Modes are Standing Waves

Let $\vec{E}(z,t) = \vec{\varepsilon}_x E_x(z,t)$ **and expand**

fiducial mass

$$\text{(7)}\quad E_x(z,t) = \sum_j A_j q_j(t) \sin(k_j z), \quad A_j = \sqrt{\frac{2\omega_j m_j}{\varepsilon_0 V}}$$

MEq (4) w/no charges

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}_\perp(\vec{r},t) = \vec{\varepsilon}_x \boxed{\frac{1}{c^2}\sum_j A_j \dot{q}_j(t)\sin(k_j t)}$$

$$= \vec{\varepsilon}_x\left(\cancel{\frac{\partial B_z}{\partial y}} - \frac{\partial B_y}{\partial z}\right) = -\vec{\varepsilon}_x \boxed{\frac{\partial B_y}{\partial z}}$$

$\mathcal{B}$ **transverse** ⇨ $B_z = 0$

From Eq. (5a) we see that

$$\vec{B} \perp \vec{E}, \vec{\varepsilon}_z \;⇨\; \vec{B}(z,t) = \vec{\varepsilon}_y B_y(z,t)$$

Putting this together we get

$$\frac{\partial B_y}{\partial z} = -\sum_j \frac{A_j}{c^2} q_j(t)\sin(k_j z)$$

$$\text{(8)}\quad B_y(z,t) = \sum_j \frac{A_j}{k_j c^2}\dot{q}_j(t)\cos(k_j z)$$