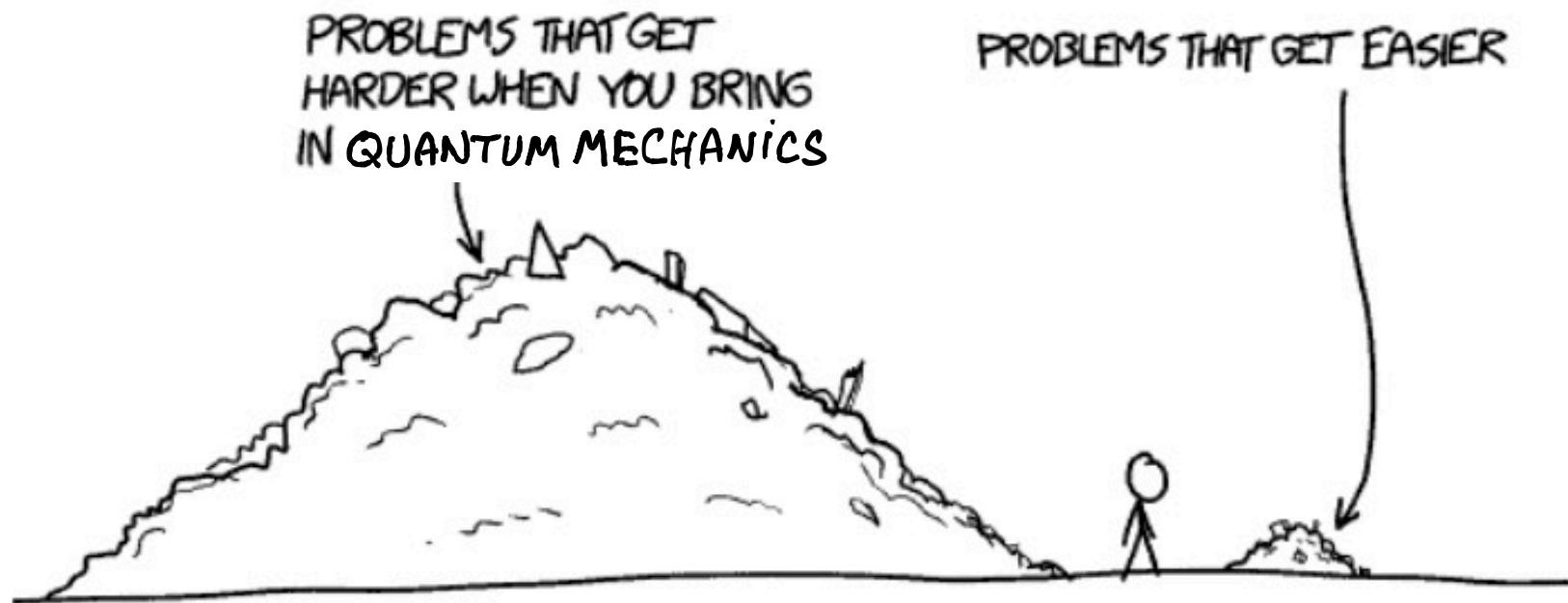


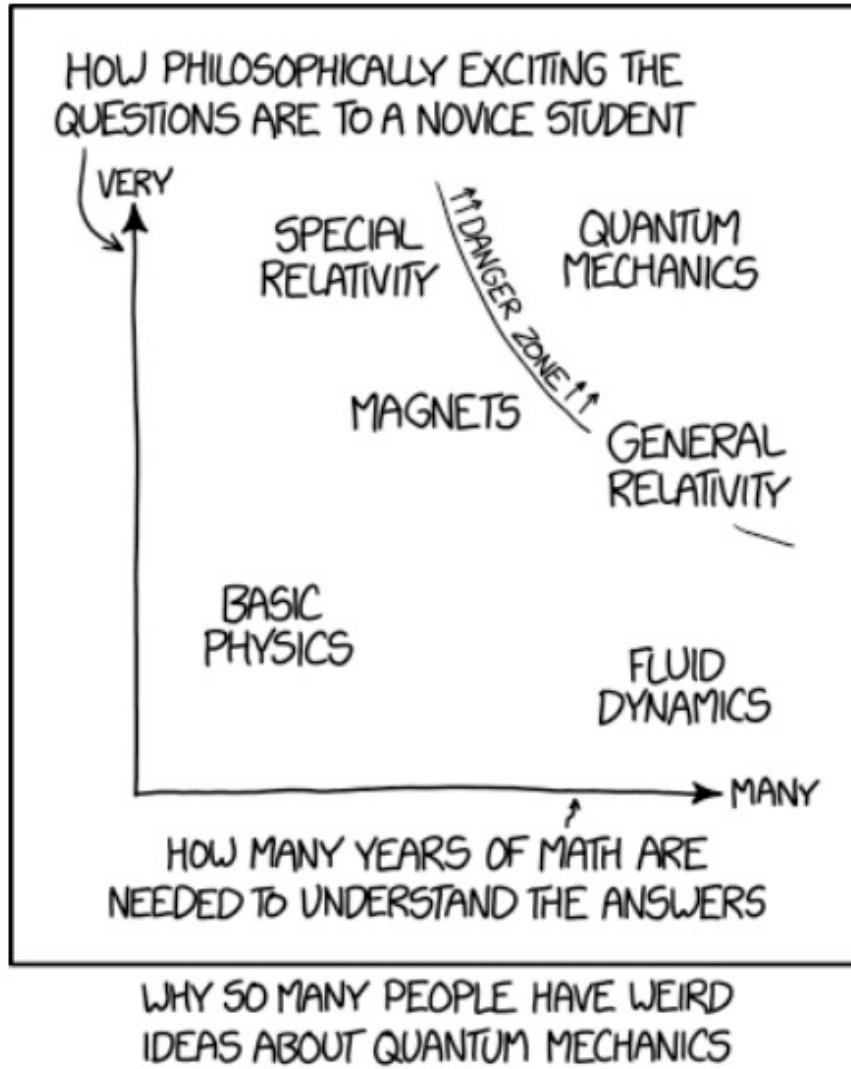
Quantum Electrodynamics – Intro to Field Theory



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Quantum Electrodynamics – Intro to Field Theory

QUANTUM



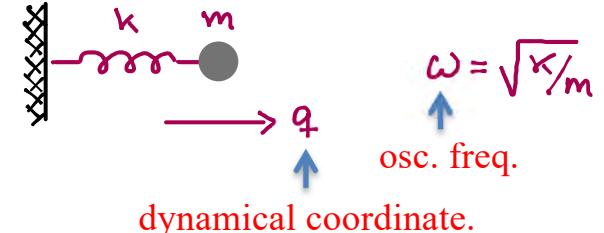
Quantum Electrodynamics – Intro to Field Theory

- (*) Primary goal of OPTI 544:
Quantum description of EM field
- (*) Challenge: 1st semester Grad level QM
(OPTI 570) does not tell how to do this
- (*) Warm-up: Quantum field theory for
vibrations (sound) in elastic rod
- (*) This is in part a review of the classical
Lagrange/Hamilton-Jacobi description
of continuous systems
- (*) Here we present the formalism as a
Cookbook Recipe for how we get from
Classical to Quantum Physics

See, e. g., Cohen-Tannoudji Vol. 2,
Appendix III, Sections 1-3.

Classical Simple Harmonic Oscillator (SHO)

Particle on
a spring



Kinetic Energy: $T = \frac{1}{2} m \dot{q}^2$

Potential Energy: $V = \frac{1}{2} k q^2 = \frac{1}{2} m \omega^2 q^2$

Lagrangian: $\mathcal{L} = T - V = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2$

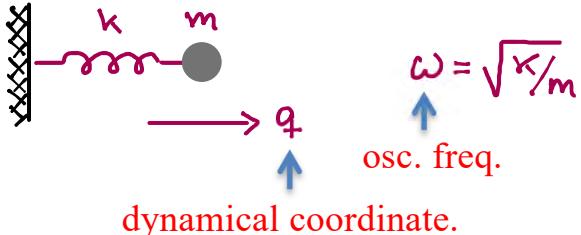
$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad \Rightarrow \quad \ddot{q} + \omega^2 q = 0$$

usual eq. of motion

Quantum Electrodynamics – Intro to Field Theory

Classical Simple Harmonic Oscillator (SHO)

Particle on a spring



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usual eq. of motion

Conjugate momentum

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m \dot{q}$$

Hamiltonian

$$\mathcal{H} = T(\dot{q} = p/m) + V(q) = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2$$

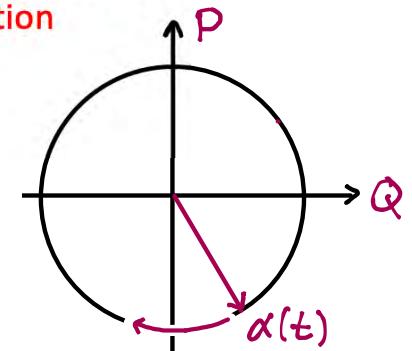
$$\left. \begin{aligned} \dot{q} &= \frac{\partial \mathcal{L}}{\partial p} = p/m \\ \dot{p} &= -\frac{\partial \mathcal{L}}{\partial q} = -m \omega^2 q \end{aligned} \right\} \quad \Rightarrow \quad \ddot{q} + \omega^2 q = 0$$

Phase plane

Scaled variables

$$Q \equiv q/q_0, \quad P \equiv p/p_0$$

$$\alpha = Q + iP \quad \left\{ \begin{array}{l} Q = \text{Re}[\alpha] \\ P = \text{Im}[\alpha] \\ \mathcal{H} = E_0 \alpha^* \alpha \end{array} \right.$$



Quantum Electrodynamics – Intro to Field Theory

Conjugate momentum

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q}$$

Hamiltonian

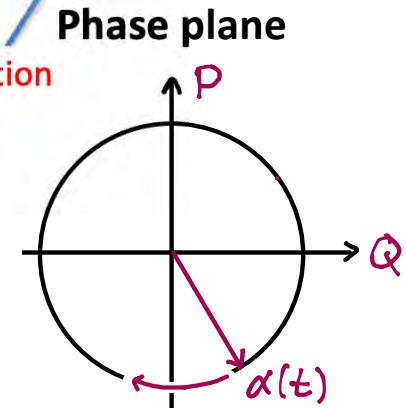
$$\mathcal{H} = T(\dot{q} = p/m) + V(q) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

$$\left. \begin{aligned} \dot{q} &= \frac{\partial \mathcal{L}}{\partial p} = p/m \\ \dot{p} &= -\frac{\partial \mathcal{L}}{\partial q} = -m\omega^2 q \end{aligned} \right\} \rightarrow \ddot{q} + \omega^2 q = 0$$

Scaled variables

$$Q \equiv q/q_0, \quad P \equiv p/p_0$$

$$\alpha = Q + iP \quad \left\{ \begin{aligned} Q &= \text{Re}[\alpha] \\ P &= \text{Im}[\alpha] \\ \mathcal{H} &= E_0 \alpha^* \alpha \end{aligned} \right.$$



solution

Quantum Harmonic Oscillator

Formal Quantization Procedure:

$$q \rightarrow \hat{q}, \quad p \rightarrow \hat{p}, \quad [\hat{q}, \hat{p}] = i\hbar$$

Choose $E_0 = \hbar\omega \rightarrow q_0 = \sqrt{\frac{2\hbar}{m\omega}}, \quad p_0 = \sqrt{2m\hbar\omega}$

natural scale

$$\alpha \rightarrow \hat{\alpha} = \hat{Q} + i\hat{P} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{q} + i\frac{\hat{p}}{m\omega} \right)$$

$$[\hat{\alpha}, \hat{\alpha}^\dagger] = 1$$

Rewrite:

$$\hat{H} = \hbar\omega(\hat{Q}^2 + \hat{P}^2) = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2})$$

$$\hat{N} = \hat{a}^\dagger \hat{a} \quad (\text{number operator})$$

Quantum Electrodynamics – Intro to Field Theory

Quantum Harmonic Oscillator

Formal Quantization Procedure:

$$q \rightarrow \hat{q}, \quad p \rightarrow \hat{p}, \quad [\hat{q}, \hat{p}] = i\hbar$$

Choose $E_0 = \hbar\omega$ \uparrow
natural scale \downarrow

$$\hat{q}_0 = \sqrt{\frac{2\hbar}{m\omega}}, \quad \hat{p}_0 = \sqrt{2m\hbar\omega}$$

$$\alpha \rightarrow \hat{a} = \hat{Q} + i\hat{P} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{q} + i\frac{\hat{p}}{m\omega} \right)$$
$$[\hat{a}, \hat{a}^\dagger] = 1$$

Rewrite:

$$\hat{H} = \hbar\omega(\hat{Q}^2 + \hat{P}^2) = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2})$$
$$\hat{N} = \hat{a}^\dagger \hat{a} \quad (\text{number operator})$$

Commutator $[\hat{H}, \hat{N}] = 0$

joint energy/number states $|n\rangle$

$$\hat{H}|n\rangle = \hbar\omega(n + \frac{1}{2})|n\rangle$$
$$\hat{N}|n\rangle = n|n\rangle$$

Commutators

$$\begin{aligned} [\hat{N}, \hat{a}^\dagger] &= \hat{a}^\dagger \\ [\hat{N}, \hat{a}] &= -\hat{a} \end{aligned} \quad \left. \right\} \quad \Rightarrow$$

$$\begin{aligned} \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \\ \hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \\ \hat{a}|0\rangle &= 0 \end{aligned}$$

Generating excited states

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle$$

Quantum Electrodynamics – Intro to Field Theory

Commutator $[\hat{H}, \hat{N}] = 0$

→ joint energy/number states $|n\rangle$

$$\hat{H}|n\rangle = \hbar\omega(n + 1/2)|n\rangle$$

$$\hat{N}|n\rangle = n|n\rangle$$

Commutators

$$\begin{aligned} [\hat{N}, \hat{a}^+] &= \hat{a}^+ \\ [\hat{N}, \hat{a}] &= -\hat{a} \end{aligned} \quad \left. \right\}$$

$$\begin{aligned} \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \\ \hat{a}^+|n\rangle &= \sqrt{n+1}|n+1\rangle \\ \hat{a}|0\rangle &= 0 \end{aligned}$$

Generating excited states

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^+)^n |0\rangle$$

Expectation values for \hat{q} and \hat{p} in number states

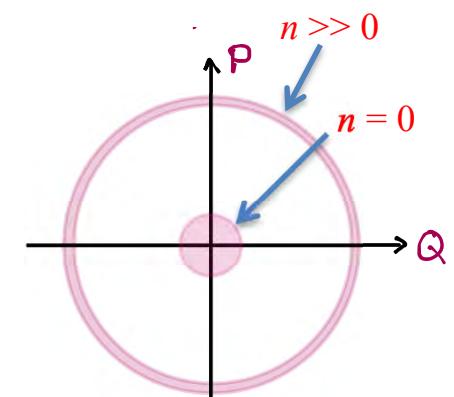
$$\langle n|\hat{q}|n\rangle = \langle n|\hat{p}|n\rangle = 0$$

$$\langle n|\hat{q}^2|n\rangle = \frac{q_0^2}{2}(n + 1/2) \neq 0$$

$$\langle n|\hat{p}^2|n\rangle = \frac{p_0^2}{2}(n + 1/2) \neq 0$$

$$\Delta q \Delta p = \frac{q_0 p_0}{2}(n + 1/2) = \hbar(n + 1/2)$$

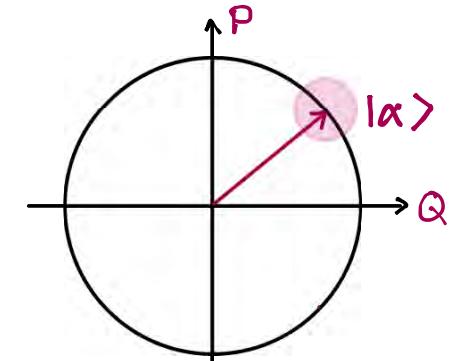
Phase space visualization
of number states



Quasi-classical
(coherent) state

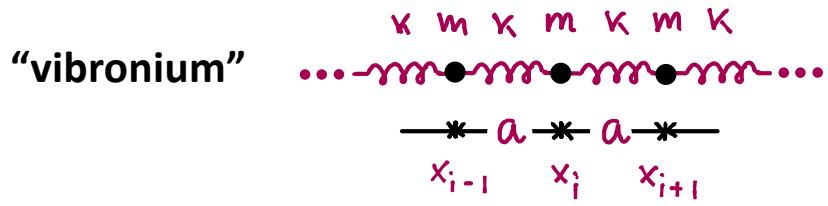
$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_i \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$\Delta q \Delta p = \hbar/2, \quad \Delta Q = \Delta P$$



Quantum Electrodynamics – Intro to Field Theory

Lagrange formulation of 1D Scalar Field



Configuration space = $\{x_i\}$ (set of N osc. positions)

$$T = \sum_{i=1}^N \frac{1}{2} m \dot{x}_i^2, \quad V = \sum_{i=1}^N \frac{1}{2} K (x_{i+1} - x_i)^2$$



Lagrangian, equations of motion

Continuum limit \rightarrow Elastic rod

$$N \rightarrow \infty \quad m/a \rightarrow \mu \quad \leftarrow \text{linear mass density}$$

$$a \rightarrow dx \quad \nu a \rightarrow Y \quad \leftarrow \text{Youngs modulus}$$

$$\{x_i\} \rightarrow \eta(x) \quad \leftarrow \text{displacement field (sound)}$$

Rewrite

$$T = \lim_{N \rightarrow \infty} \sum_{i=1}^N a \frac{1}{2} \left(\frac{m}{a} \right) \dot{x}_i^2 = \int dx \frac{1}{2} \mu \left(\frac{\partial \eta}{\partial t} \right)^2$$

$$V = \lim_{N \rightarrow \infty} \sum_{i=1}^N a \frac{1}{2} \nu a \left(\frac{x_{i+1} - x_i}{a} \right)^2 = \int dx \frac{1}{2} Y \left(\frac{\partial \eta}{\partial x} \right)^2$$

Lagrangian:

$$\mathcal{L} = T - V = \int dx \frac{1}{2} \mu \left(\frac{\partial \eta}{\partial t} \right)^2 - \int dx \frac{1}{2} Y \left(\frac{\partial \eta}{\partial x} \right)^2$$

Notes, Homework \rightarrow Scalar wave equation

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{Y}{\mu} \frac{\partial^2 \eta}{\partial x^2} = 0$$

– Not yet ready for Quantization –

Quantum Electrodynamics – Intro to Field Theory

Rewrite

$$T = \lim_{N \rightarrow \infty} \sum_{i=1}^N a \frac{1}{2} \left(\frac{m}{a} \dot{x}_i \right)^2 = \int dx \frac{1}{2} \mu \left(\frac{\partial \eta}{\partial t} \right)^2$$

$$V = \lim_{N \rightarrow \infty} \sum_{i=1}^N a \frac{1}{2} \times a \left(\frac{x_{i+1} - x_i}{a} \right)^2 = \int dx \frac{1}{2} \gamma \left(\frac{\partial \eta}{\partial x} \right)^2$$

Lagrangian:

$$\mathcal{L} = T - V = \int dx \frac{1}{2} \mu \left(\frac{\partial \eta}{\partial t} \right)^2 - \int dx \frac{1}{2} \gamma \left(\frac{\partial \eta}{\partial x} \right)^2$$

Notes, Homework → Scalar wave equation

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\gamma}{\mu} \frac{\partial^2 \eta}{\partial x^2} = 0$$

– Not yet ready for Quantization –

Normal Mode Decomposition

Field in cavity:



Solutions to wave eq.

Let $\eta(x, t) = g(t)u(x) = g_0 e^{i\omega t} u(x)$ ↗

$$\ddot{u} - \nu^2 u'' = -\omega^2 g(t) u(x) - \nu^2 g(t) u''(x) = 0$$



$$u''(x) = -k^2 u(x), \quad k = \omega/\nu$$

Solutions in cavity:

$$u_{kx}(x) = \sqrt{\frac{2}{L}} \sin(kx), \quad k = \frac{n\pi}{L}$$

These standing waves are a set of Normal Modes

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Normal Mode Decomposition

Field in cavity:



Solutions to wave eq.

Let $y(x,t) = g(t)u(x) = g_0 e^{i\omega t} u(x)$

$$i\ddot{y} - \nu^2 y'' = -\omega^2 g(t) u(x) - \nu^2 g(t) u''(x) = 0$$



$$u''(x) = -k^2 u(x), \quad k = \omega/\nu$$

Solutions in cavity:

$$u_{nk}(x) = \sqrt{\frac{2}{L}} \sin(kx), \quad k = \frac{n\pi}{L}$$

These standing waves are a set of Normal Modes

These modes are orthonormal and complete



$$y(x,t) = \sqrt{L} \sum_{nk} q_{nk}(t) u_{nk}(x)$$

Normal mode expansion of $y(x,t)$ in basis $u_{nk}(x)$

Lagrangian for the acoustic field:

$$\begin{aligned} T &= \int dx \frac{1}{2} \mu \left(\frac{\partial y}{\partial t} \right)^2 = \sum_{nk} \underbrace{\frac{1}{2} \mu L q_{nk} \dot{q}_{nk}}_M \underbrace{\int dx u_{nk}(x) u_{nk}(x)}_{\delta_{nk}} \\ &= \sum_{nk} \frac{1}{2} M \dot{q}_{nk}^2 \end{aligned}$$

$$\begin{aligned} V &= \int dx \frac{1}{2} \gamma \left(\frac{\partial y}{\partial x} \right)^2 = \sum_{nk} \underbrace{\frac{1}{2} \gamma L q_{nk} q_{nk}}_M \underbrace{\int dx \left(\frac{\partial u_{nk}}{\partial x} \right) \left(\frac{\partial u_{nk}}{\partial x} \right)}_{\delta_{nk}} \\ &= \sum_{nk} \frac{1}{2} M \omega_{nk}^2 q_{nk}^2 \end{aligned}$$



$$\mathcal{L} = T - V = \sum_{nk} \left(\frac{1}{2} M \dot{q}_{nk}^2 - \frac{1}{2} M \omega_{nk}^2 q_{nk}^2 \right) = \sum_{nk} \mathcal{L}_{nk}$$