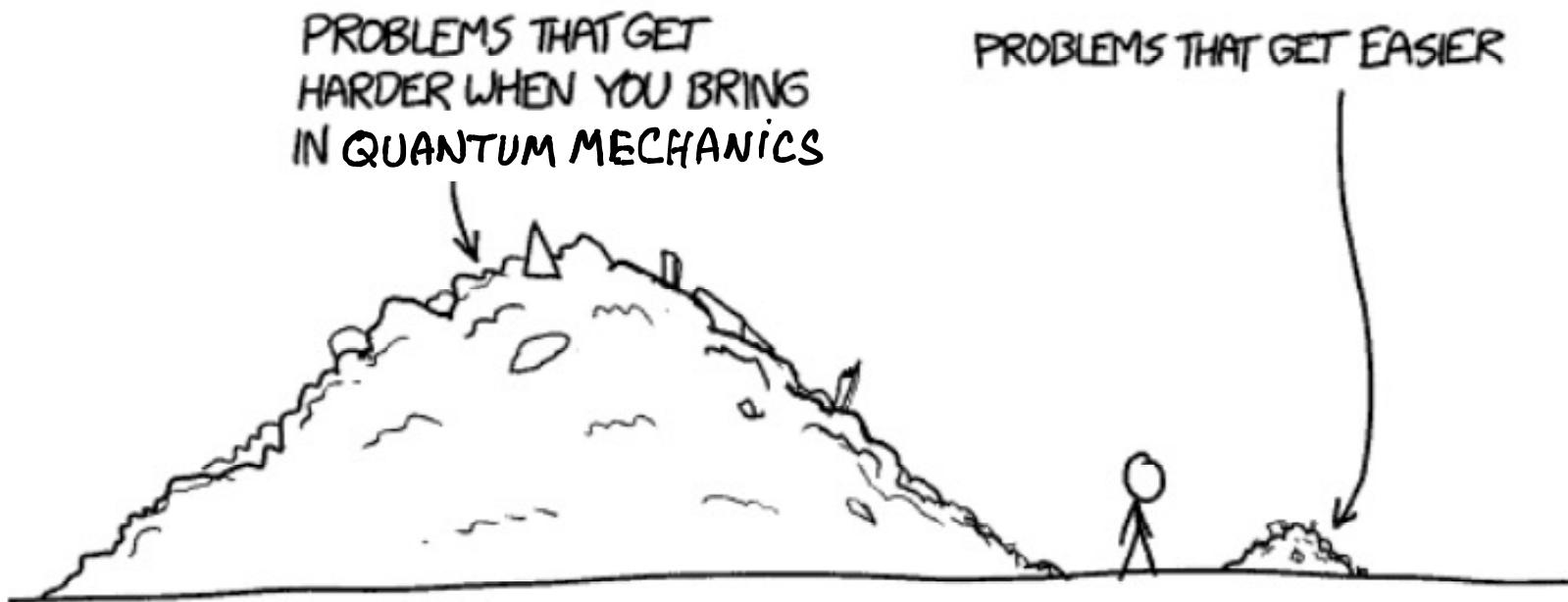
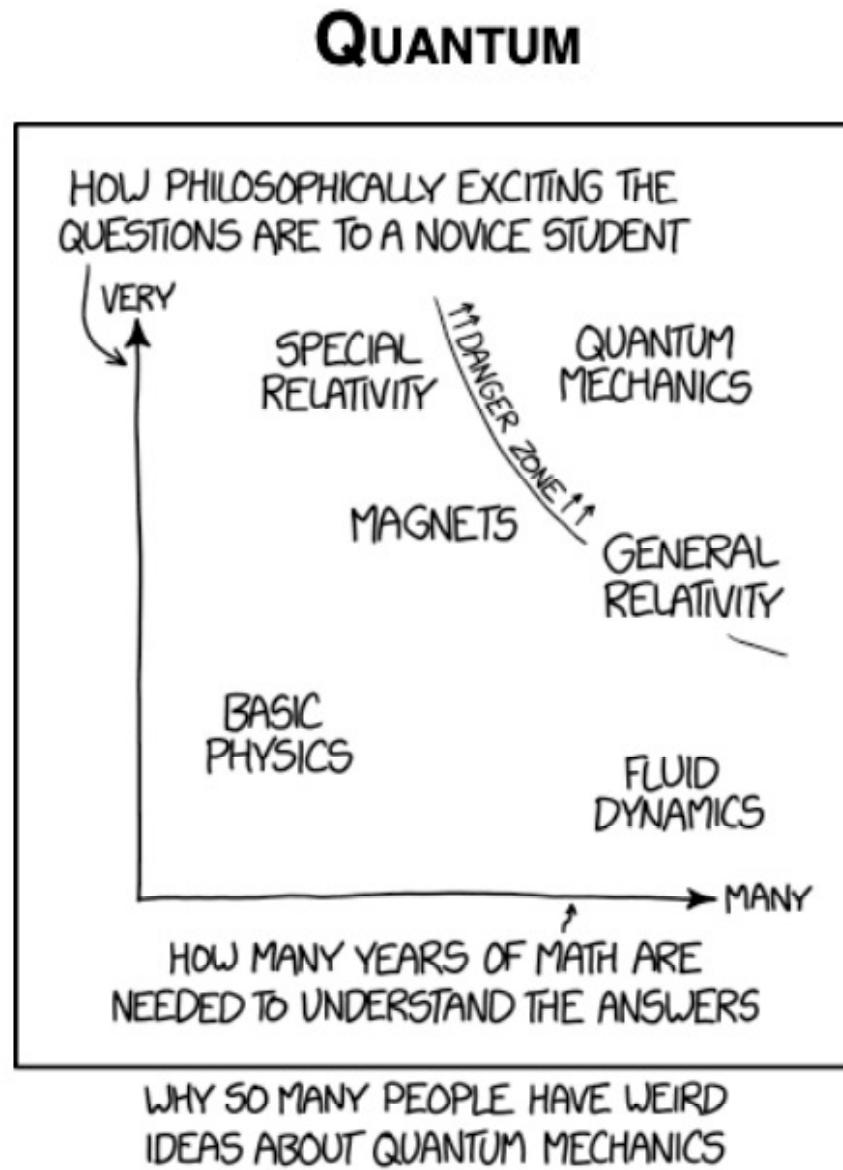


Quantum Electrodynamics – Intro to Field Theory



Source: xkcd.com

Quantum Electrodynamics – Intro to Field Theory



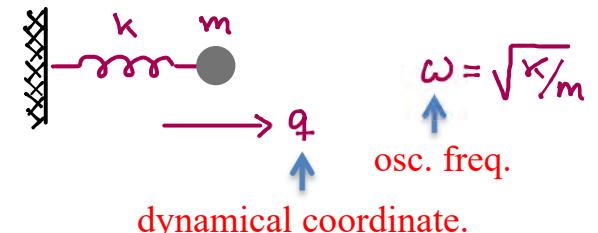
Quantum Electrodynamics – Intro to Field Theory

- (*) Primary goal of OPTI 544:
Quantum description of EM field
- (*) Challenge: 1st semester Grad level QM
(OPTI 570) does not tell how to do this
- (*) Warm-up: Quantum field theory for vibrations (sound) in elastic rod
- (*) This is in part a review of the classical Lagrange/Hamilton-Jacobi description of continuous systems
- (*) Here we present the formalism as a Cookbook Recipe for how we get from Classical to Quantum Physics

See, e. g., Cohen-Tannoudji Vol. 2,
Appendix III, Sections 1-3.

Classical Simple Harmonic Oscillator (SHO)

Particle on a spring



Kinetic Energy: $T = \frac{1}{2} m \dot{q}^2$

Potential Energy: $V = \frac{1}{2} k q_f^2 = \frac{1}{2} m \omega^2 q^2$

Lagrangian: $\mathcal{L} = T - V = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2$

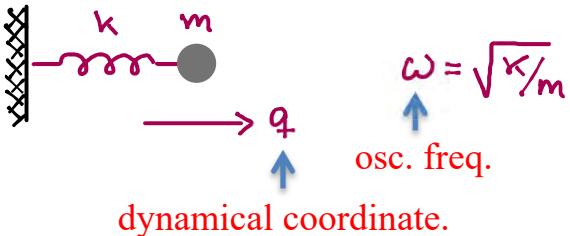
$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad \Rightarrow \quad \ddot{q} + \omega^2 q = 0$$

usual eq. of motion

Quantum Electrodynamics – Intro to Field Theory

Classical Simple Harmonic Oscillator (SHO)

Particle on a spring



Kinetic Energy: $T = \frac{1}{2} m \dot{q}^2$

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Lagrangian: $\mathcal{L} = T - V = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2$

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usual eq. of motion

Conjugate momentum

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m \dot{q}$$

Hamiltonian

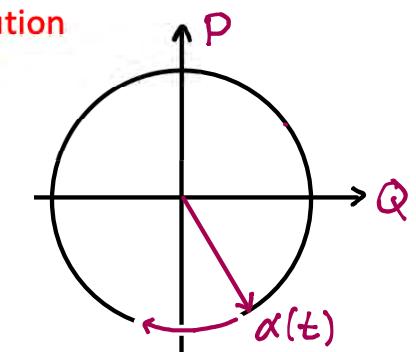
$$\mathcal{H} = T(\dot{q}) + V(q) = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2$$

$$\begin{aligned} \dot{q} &= \frac{\partial \mathcal{H}}{\partial p} = p/m \\ \dot{p} &= -\frac{\partial \mathcal{H}}{\partial q} = -m \omega^2 q \end{aligned} \quad \left. \right\} \quad \ddot{q} + \omega^2 q = 0$$

Phase plane
Scaled variables

$$Q \equiv q/q_0, \quad P \equiv p/p_0$$

$$\alpha = Q + iP \quad \left\{ \begin{array}{l} Q = \text{Re}[\alpha] \\ P = \text{Im}[\alpha] \\ \mathcal{H} = E_0 \alpha^* \alpha \end{array} \right.$$



Quantum Electrodynamics – Intro to Field Theory

Conjugate momentum

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q}$$

Hamiltonian

$$\mathcal{H} = T(\dot{q} = p/m) + V(q) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

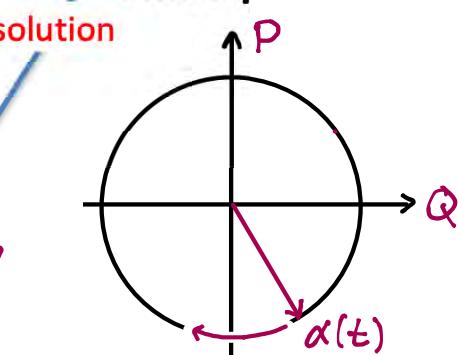
$$\begin{aligned}\dot{q} &= \frac{\partial \mathcal{L}}{\partial p} = p/m \\ \dot{p} &= -\frac{\partial \mathcal{L}}{\partial q} = -m\omega^2 q\end{aligned}$$

$$\ddot{q} + \omega^2 q = 0$$

Scaled variables

$$Q \equiv q/q_0, \quad P \equiv p/p_0$$

$$\alpha = Q + iP \quad \left\{ \begin{array}{l} Q = \text{Re}[\alpha] \\ P = \text{Im}[\alpha] \\ \mathcal{H} = E_0 \alpha^* \alpha \end{array} \right.$$



solution

Quantum Harmonic Oscillator

Formal Quantization Procedure:

$$q \rightarrow \hat{q}, \quad p \rightarrow \hat{p}, \quad [\hat{q}, \hat{p}] = i\hbar$$

Choose $E_0 = \hbar\omega \quad \Rightarrow \quad q_0 = \sqrt{\frac{2\hbar}{m\omega}}, \quad p_0 = \sqrt{2m\hbar\omega}$

natural scale

$$\alpha \rightarrow \hat{\alpha} = \hat{Q} + i\hat{P} = \sqrt{\frac{m\omega}{2\hbar}} (\hat{q} + i\frac{\hat{p}}{m\omega})$$

$$[\hat{\alpha}, \hat{\alpha}^\dagger] = 1$$

Rewrite:

$$\hat{H} = \hbar\omega(\hat{Q}^2 + \hat{P}^2) = \hbar\omega(\hat{\alpha}^\dagger \hat{\alpha} + \frac{1}{2})$$

$$\hat{N} = \hat{\alpha}^\dagger \hat{\alpha} \quad (\text{number operator})$$

Quantum Electrodynamics – Intro to Field Theory

Quantum Harmonic Oscillator

Formal Quantization Procedure:

$$q \rightarrow \hat{q}, \quad p \rightarrow \hat{p}, \quad [\hat{q}, \hat{p}] = i\hbar$$

Choose $E_0 = \frac{\hbar\omega}{2}$ \downarrow
natural scale

$$\hat{q}_0 = \sqrt{\frac{2\hbar}{m\omega}}, \quad \hat{p}_0 = \sqrt{2m\hbar\omega}$$

$$\alpha \rightarrow \hat{a} = \hat{Q} + i\hat{P} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{q} + i\frac{\hat{p}}{m\omega} \right)$$
$$[\hat{a}, \hat{a}^\dagger] = 1$$

Rewrite:

$$\hat{H} = \hbar\omega(\hat{Q}^2 + \hat{P}^2) = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2})$$
$$\hat{N} = \hat{a}^\dagger \hat{a} \quad (\text{number operator})$$

Commutator $[\hat{H}, \hat{N}] = 0$

joint energy/number states $|n\rangle$

$$\hat{H}|n\rangle = \hbar\omega(n + \frac{1}{2})|n\rangle$$
$$\hat{N}|n\rangle = n|n\rangle$$

Commutators

$$\begin{aligned} [\hat{N}, \hat{a}^\dagger] &= \hat{a}^\dagger \\ [\hat{N}, \hat{a}] &= -\hat{a} \end{aligned} \quad \left. \right\} \quad \downarrow$$

$$\begin{aligned} \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \\ \hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \\ \hat{a}|0\rangle &= 0 \end{aligned}$$

Generating excited states

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle$$

Quantum Electrodynamics – Intro to Field Theory

Commutator $[\hat{H}, \hat{N}] = 0$

→ joint energy/number states $|n\rangle$

$$\hat{H}|n\rangle = \hbar\omega(n + 1/2)|n\rangle$$

$$\hat{N}|n\rangle = n|n\rangle$$

Commutators

$$\begin{aligned} [\hat{N}, \hat{a}^+] &= \hat{a}^+ \\ [\hat{N}, \hat{a}] &= -\hat{a} \end{aligned} \quad \left. \right\}$$

$$\begin{aligned} \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \\ \hat{a}^+|n\rangle &= \sqrt{n+1}|n+1\rangle \\ \hat{a}|0\rangle &= 0 \end{aligned}$$

Generating excited states

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^+)^n |0\rangle$$

Expectation values for \hat{q} and \hat{p} in number states

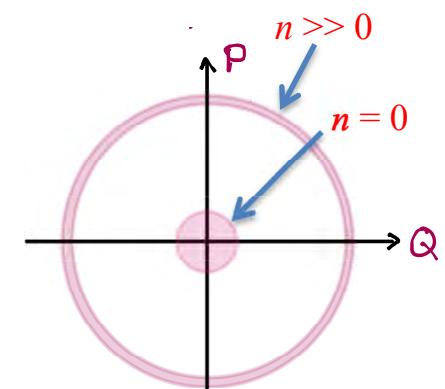
$$\langle n|\hat{q}|n\rangle = \langle n|\hat{p}|n\rangle = 0$$

$$\langle n|\hat{q}^2|n\rangle = \frac{q_0^2}{2}(n + 1/2) \neq 0$$

$$\langle n|\hat{p}^2|n\rangle = \frac{p_0^2}{2}(n + 1/2) \neq 0$$

$$\Delta q \Delta p = \frac{q_0 p_0}{2}(n + 1/2) = \hbar(n + 1/2)$$

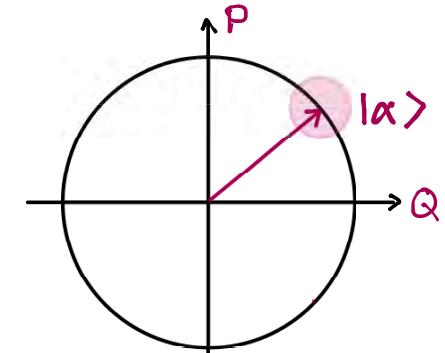
Phase space visualization
of number states



Quasi-classical
(coherent) state

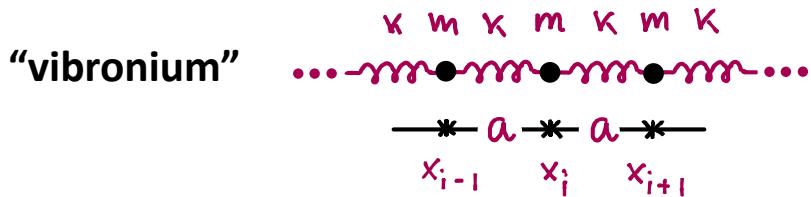
$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_i \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$\Delta q \Delta p = \hbar/2, \quad \Delta Q = \Delta P$$



Quantum Electrodynamics – Intro to Field Theory

Lagrange formulation of 1D Scalar Field



Configuration space = $\{x_i\}$ (set of N osc. positions)



$$T = \sum_{i=1}^N \frac{1}{2} m \dot{x}_i^2, \quad V = \sum_{i=1}^N \frac{1}{2} k (x_{i+1} - x_i)^2$$



Lagrangian, equations of motion

Continuum limit \rightarrow Elastic rod

$$N \rightarrow \infty \quad m/a \rightarrow \mu \quad \leftarrow \text{linear mass density}$$

$$a \rightarrow dx \quad k a \rightarrow Y \quad \leftarrow \text{Youngs modulus}$$

$$\{x_i\} \rightarrow \eta(x) \quad \leftarrow \text{displacement field (sound)}$$

Rewrite

$$T = \lim_{N \rightarrow \infty} \sum_{i=1}^N \frac{1}{2} \left(\frac{m}{a} \right) \dot{x}_i^2 = \int dx \frac{1}{2} \mu \left(\frac{\partial \eta}{\partial t} \right)^2$$

$$V = \lim_{N \rightarrow \infty} \sum_{i=1}^N \frac{1}{2} k a \left(\frac{x_{i+1} - x_i}{a} \right)^2 = \int dx \frac{1}{2} Y \left(\frac{\partial \eta}{\partial x} \right)^2$$

Lagrangian:

$$\mathcal{L} = T - V = \int dx \frac{1}{2} \mu \left(\frac{\partial \eta}{\partial t} \right)^2 - \int dx \frac{1}{2} Y \left(\frac{\partial \eta}{\partial x} \right)^2$$

Notes, Homework \rightarrow Scalar wave equation

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{Y}{\mu} \frac{\partial^2 \eta}{\partial x^2} = 0$$

– Not yet ready for Quantization –

Quantum Electrodynamics – Intro to Field Theory

Rewrite

$$T = \lim_{N \rightarrow \infty} \sum_{i=1}^N a \frac{1}{2} \left(\frac{m}{a} \dot{x}_i \right)^2 = \int dx \frac{1}{2} \mu \left(\frac{\partial \eta}{\partial t} \right)^2$$

$$V = \lim_{N \rightarrow \infty} \sum_{i=1}^N a \frac{1}{2} \kappa a \left(\frac{x_{i+1} - x_i}{a} \right)^2 = \int dx \frac{1}{2} \gamma \left(\frac{\partial \eta}{\partial x} \right)^2$$

Lagrangian:

$$\mathcal{L} = T - V = \int dx \frac{1}{2} \mu \left(\frac{\partial \eta}{\partial t} \right)^2 - \int dx \frac{1}{2} \gamma \left(\frac{\partial \eta}{\partial x} \right)^2$$

Notes, Homework → Scalar wave equation

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\gamma}{\mu} \frac{\partial^2 \eta}{\partial x^2} = 0$$

– Not yet ready for Quantization –

Normal Mode Decomposition

Field in cavity:



Solutions to wave eq.

Let $\eta(x, t) = g(t)u(x) = g_0 e^{i\omega t} u(x)$ ↗
 $\ddot{\eta} - \omega^2 \eta'' = -\omega^2 g(t) u(x) - \omega^2 g(t) u''(x) = 0$



$$u''(x) = -\frac{\omega^2}{\mu} u(x), \quad \frac{\omega^2}{\mu} = \omega^2 / \gamma$$

Solutions in cavity:

$$u_{\ell k}(x) = \sqrt{\frac{2}{L}} \sin(\ell k x), \quad \ell k = \frac{n\pi}{L}$$

These standing waves are a set of Normal Modes

Quantum Electrodynamics – Intro to Field Theory

Normal Mode Decomposition

Field in cavity:



Solutions to wave eq.

$$\text{Let } y(x,t) = g(t)u(x) = g_0 e^{i\omega t} u(x)$$

$$i\ddot{y} - \nu^2 y'' = -\omega^2 g(t) u(x) - \nu^2 g(t) u''(x) = 0$$

$$u''(x) = -k^2 u(x), \quad k = \omega/\nu$$

Solutions in cavity:

$$u_{nk}(x) = \sqrt{\frac{2}{L}} \sin(k_n x), \quad k_n = \frac{n\pi}{L}$$

These standing waves are a set of Normal Modes

These modes are orthonormal and complete



$$y(x,t) = \sqrt{L} \sum_{nk} q_{nk}(t) u_{nk}(x)$$

Normal mode expansion of $y(x,t)$ in basis $u_{nk}(x)$

Lagrangian for the acoustic field:

$$T = \int dx \frac{1}{2} \mu \left(\frac{\partial y}{\partial t} \right)^2 = \sum_{nk, n'k'} \underbrace{\frac{1}{2} \mu L q_{nk} \dot{q}_{nk}}_{M} \underbrace{\int dx u_{nk}(x) u_{n'k'}(x)}_{\delta_{nk} \delta_{n'k'}}$$

$$V = \int dx \frac{1}{2} \gamma \left(\frac{\partial y}{\partial x} \right)^2 = \sum_{nk, n'k'} \underbrace{\frac{1}{2} \gamma L q_{nk} q_{n'k'}}_{M \omega_{nk}^2} \underbrace{\int dx \left(\frac{\partial u_{nk}}{\partial x} \right) \left(\frac{\partial u_{n'k'}}{\partial x} \right)}_{\delta_{nk} \delta_{n'k'}}$$



$$\mathcal{L} = T - V = \sum_{nk} \left(\frac{1}{2} M \dot{q}_{nk}^2 - \frac{1}{2} M \omega_{nk}^2 q_{nk}^2 \right) = \sum_{nk} \mathcal{L}_{nk}$$

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The rest now follows from the Lagrangian

$$\mathcal{L} = T - V = \sum_k \left(\frac{1}{2} M \dot{q}_k^2 - \frac{1}{2} M \omega_k^2 q_k^2 \right) = \sum_k \mathcal{L}_k$$



Canonical Momentum

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = M \dot{q}_k$$

Hamiltonian

$$\mathcal{H}(\{p_k, q_k\}) = T + V = \sum_k \left(\frac{p_k^2}{2M} + \frac{1}{2} M \omega_k^2 q_k^2 \right)$$

(collection of SHO's, one for each normal mode)

Following the standard recipe...

$$E_{0,k} = \hbar \omega_k, \quad q_{0,k} = \sqrt{2\hbar/M\omega_k}, \quad p_{0,k} = \sqrt{2M\hbar\omega_k}$$

$$Q_k = q_k/q_{0,k}, \quad P_k = p_k/p_{0,k}, \quad \alpha_k = Q_k + iP_k$$

... we get solutions

$$\alpha_k(t) = Q_k(t) + iP_k(t) = \alpha_k(0) e^{-i\omega_k t}$$

This finally gives us

$$\mathcal{H} = \sum_k \hbar \omega_k (Q_k^2 + P_k^2) = \sum_k \hbar \omega_k \alpha_k^* \alpha_k$$

$$\begin{aligned} y(x, t) &= \sqrt{L} \sum_k q_{0,k} \alpha_k(t) u_k(x) \\ &= \frac{1}{2} \sum_k \sqrt{L q_{0,k}^2} (\alpha_k(t) u_k(x) + \alpha_k^*(t) u_k^*(x)) \end{aligned}$$

$$\alpha = Q + iP \quad Q = \frac{1}{2}(\alpha + \alpha^*) \quad Q = \frac{q_0}{2}(\alpha + \alpha^*)$$

$$\alpha^* = Q - iP \quad P = \frac{1}{2i}(\alpha - \alpha^*) \quad P = \frac{p_0}{2}(\alpha - \alpha^*)$$

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... we get solutions

$$\alpha_{\mathbf{k}}(t) = Q_{\mathbf{k}}(t) + i P_{\mathbf{k}}(t) = \alpha_{\mathbf{k}}(0) e^{-i\omega_{\mathbf{k}} t}$$

This finally gives us

$$\mathcal{H} = \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} (Q_{\mathbf{k}}^2 + P_{\mathbf{k}}^2) = \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} \alpha_{\mathbf{k}}^* \alpha_{\mathbf{k}}$$

$$\begin{aligned} g(x, t) &= \sqrt{L} \sum_{\mathbf{k}} q_{\mathbf{k}}(t) u_{\mathbf{k}}(x) \\ &= \frac{1}{2} \sum_{\mathbf{k}} \sqrt{L q_{0, \mathbf{k}}^2} (\alpha_{\mathbf{k}}(t) u_{\mathbf{k}}(x) + \alpha_{\mathbf{k}}^*(t) u_{\mathbf{k}}(x)) \end{aligned}$$

Formal Quantization Procedure:

$$\begin{aligned} q_{\mathbf{k}} &\rightarrow \hat{q}_{\mathbf{k}}, \quad p_{\mathbf{k}} \rightarrow \hat{p}_{\mathbf{k}}, \quad \alpha_{\mathbf{k}} \rightarrow \hat{\alpha}_{\mathbf{k}} \\ [\hat{q}_{\mathbf{k}}, \hat{p}_{\mathbf{k}'}] &= i \hbar \delta_{\mathbf{k}, \mathbf{k}'}, \quad [\hat{\alpha}_{\mathbf{k}}, \hat{\alpha}_{\mathbf{k}'}^+] = \delta_{\mathbf{k}, \mathbf{k}'}, \quad [\hat{\alpha}_{\mathbf{k}}, \hat{\alpha}_{\mathbf{k}'}] = 0 \end{aligned}$$

Note: $\mathbf{k} \neq \mathbf{k}' \rightarrow$ operators commute
(normal modes = independent degs. of freedom)

Hamiltonian & Quantized fields

$$\hat{H} = \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} (\hat{\alpha}_{\mathbf{k}}^+ \hat{\alpha}_{\mathbf{k}} + \frac{1}{2})$$

$$\hat{g}(x) = \sqrt{L} \sum_{\mathbf{k}} \hat{q}_{\mathbf{k}} u_{\mathbf{k}}(x) = \sum_{\mathbf{k}} \sqrt{L q_{0, \mathbf{k}}^2} (\hat{\alpha}_{\mathbf{k}} u_{\mathbf{k}}(x) + \hat{\alpha}_{\mathbf{k}}^+ u_{\mathbf{k}}(x))$$

$$\hat{\pi}(x) = \frac{1}{\sqrt{L}} \sum_{\mathbf{k}} \hat{p}_{\mathbf{k}} u_{\mathbf{k}}(x) = -i \sum_{\mathbf{k}} \sqrt{\frac{\hbar^2 k^2}{L}} (\hat{\alpha}_{\mathbf{k}} u_{\mathbf{k}}(x) - \hat{\alpha}_{\mathbf{k}}^+ u_{\mathbf{k}}(x))$$

field $\hat{g}(x)$ and canonical momentum field $\hat{\pi}(x)$

$$\rightarrow [\hat{g}(x), \hat{\pi}(x')] = i \hbar \delta(x - x')$$

Quantum Electrodynamics – Intro to Field Theory

Formal Quantization Procedure:

$$q_k \rightarrow \hat{q}_k, \quad p_k \rightarrow \hat{p}_k, \quad \alpha_k \rightarrow \hat{\alpha}_k$$

$$[\hat{q}_k, \hat{p}_{k'}] = i\hbar \delta_{kk'}, \quad [\hat{\alpha}_k, \hat{\alpha}^+_{k'}] = \delta_{kk'}, \quad [\hat{\alpha}_k, \hat{\alpha}_{k'}] = 0$$

Note: $k \neq k'$ → operators commute
(normal modes = independent degs. of freedom)

Hamiltonian & Quantized fields

$$\hat{H} = \sum_k \hbar \omega_k (\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2})$$

$$\hat{y}(x) = \sqrt{L} \sum_k \hat{q}_k u_k(x) = \sum_k \sqrt{\frac{\hbar \omega_k}{2m}} (\hat{a}_k u_k(x) + \hat{a}_k^\dagger u_k(x))$$

$$\hat{\pi}(x) = \frac{1}{\sqrt{L}} \sum_k \hat{p}_k u_k(x) = -i \sum_k \sqrt{\frac{\hbar \omega_k}{2m}} (\hat{a}_k u_k(x) - \hat{a}_k^\dagger u_k(x))$$

field $\hat{y}(x)$ and canonical momentum field $\hat{\pi}(x)$

$$\rightarrow [\hat{y}(x), \hat{\pi}(x')] = i\hbar \delta(x-x')$$

Quantum States:

The quantum field is a collection of QSHO's



State space = tensor product of SHO spaces

Fock Space $\mathcal{E} = \mathcal{E}_{k_1} \otimes \mathcal{E}_{k_2} \otimes \mathcal{E}_{k_3} \otimes \dots \otimes \mathcal{E}_{k_j}$ SHO space

Fock State $| \{n_{k_1}, n_{k_2}, \dots\} \rangle = |n_{k_1}\rangle \otimes |n_{k_2}\rangle \otimes \dots |n_{k_j}\rangle$

$\hat{a}_{k_i}, \hat{a}_{k_i}^\dagger$ destroy/create excitations in mode k_i

Vacuum State $|0\rangle$ ← zero quanta in every mode

Favorite Question: What is a Phonon?

Quantum Electrodynamics – Intro to Field Theory

Quantum States:

The quantum field is a collection of QSHO's



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Fock Space

$$\mathcal{E} = \mathcal{E}_{k_1} \otimes \mathcal{E}_{k_2} \otimes \mathcal{E}_{k_3} \otimes \dots, \mathcal{E}_{k_j}$$

↑
SHO space

Fock State

$$|\{n_{k_1}, n_{k_2}, \dots\}\rangle = |n_{k_1}\rangle \otimes |n_{k_2}\rangle \otimes \dots |n_{k_j}\rangle$$

↓
SHO state

$\hat{a}_{k_i}, \hat{a}_{k_i}^\dagger$ destroy/create excitations in mode k_i

Vacuum State

$$|0\rangle \quad \leftarrow \text{zero quanta in every mode}$$

Favorite Question: What is a Phonon?

Vacuum Fluctuations:

Expectation value of the Field

$$\langle 0 | \hat{\eta}(x) | 0 \rangle =$$

$$\sum_k \frac{1}{2} \sqrt{L q_{0,k}^2} (\langle 0 | \hat{a}_{k_0} | 0 \rangle u_k(x) + \langle 0 | \hat{a}_{k_0}^\dagger | 0 \rangle u_k^\dagger(x)) = 0$$

Zero Point Fluctuations

$$\langle 0 | \hat{\eta}(x)^2 | 0 \rangle$$

$$= \sum_k \frac{1}{4} L q_{0,k}^2 q_{0,k} \langle 0 | \hat{a}_{k_0}^\dagger \hat{a}_{k_0} + 1 | 0 \rangle u_k(x) u_k^\dagger(x)$$

$$= \sum_k \frac{1}{4} L q_{0,k}^2 |u_k(x)|^2 \neq 0$$

$$\sum_{kk'} \langle 0 | (\hat{a}_k u_k + \hat{a}_k^\dagger u_k^\dagger)(\hat{a}_{k'} u_{k'} + \hat{a}_{k'}^\dagger u_{k'}^\dagger) | 0 \rangle =$$

$$\sum_k \langle 0 | (\hat{a}_k \hat{a}_k^\dagger u_k u_k + \hat{a}_k \hat{a}_k^\dagger u_{k'} u_{k'} + \hat{a}_k^\dagger \hat{a}_k u_k u_{k'} + \hat{a}_k^\dagger \hat{a}_k^\dagger u_k u_{k'}) | 0 \rangle =$$

$$\sum_k \langle 0 | \hat{a}_k \hat{a}_k^\dagger u_k u_k | 0 \rangle = \sum_k \langle 0 | (\hat{a}_k^\dagger \hat{a}_k + 1) | 0 \rangle u_k u_k$$

$$\hat{\eta}(x) = \frac{1}{2} \sum_k \sqrt{L q_{0,k}^2} (\hat{a}_{k_0} u_k(x) + \hat{a}_{k_0}^\dagger u_k^\dagger(x))$$

Quantum Electrodynamics – Intro to Field Theory

Quantum States:

The quantum field is a collection of QSHO's



State space = tensor product of SHO spaces

Fock Space

$$\mathcal{E} = \mathcal{E}_{k_1} \otimes \mathcal{E}_{k_2} \otimes \mathcal{E}_{k_3} \otimes \dots, \mathcal{E}_{k_j}$$

SHO space

Fock State

$$|\{n_{k_1}, n_{k_2}, \dots\}\rangle = |n_{k_1}\rangle \otimes |n_{k_2}\rangle \otimes \dots |n_{k_j}\rangle$$

SHO state

$\hat{a}_{k_i}, \hat{a}_{k_i}^\dagger$ destroy/create excitations in mode k_i

Vacuum State

$$|0\rangle \leftarrow \text{zero quanta in every mode}$$

Favorite Question: What is a Phonon?

Vacuum Fluctuations:

Expectation value of the Field

$$\langle 0 | \hat{\eta}(x) | 0 \rangle =$$

$$\sum_k \frac{1}{2} \sqrt{\omega_{0,k}} (\langle 0 | \hat{a}_{k_0} | 0 \rangle \mu_k(x) + \langle 0 | \hat{a}_{k_0}^\dagger | 0 \rangle \mu_k^\dagger(x)) = 0$$

Zero Point Fluctuations

$$\langle 0 | \hat{\eta}(x)^2 | 0 \rangle$$

$$= \sum_k \frac{1}{4} \omega_{0,k} \langle 0 | \hat{a}_{k_0}^\dagger \hat{a}_{k_0} + 1 | 0 \rangle \mu_k(x) \mu_k(x)$$

$$= \sum_k \frac{1}{4} \omega_{0,k} |\mu_k(x)|^2 \neq 0$$

Thus $\Delta \eta(x) \neq 0$ with zero phonons in field

Note the famous divergence:

$$E_{vac} = \langle 0 | \hat{H} | 0 \rangle = \sum_k \frac{\hbar \omega_k}{2} \rightarrow \infty \text{ for } k \rightarrow \infty$$

Quantum Electrodynamics – Intro to Field Theory

Vacuum Fluctuations:

Expectation value of the Field

$$\langle 0 | \hat{\eta}(x) | 0 \rangle =$$

$$\sum_k \frac{1}{2} \sqrt{\omega_{q_0, k}} (\langle 0 | \hat{a}_{k, 0} | 0 \rangle u_k(x) + \langle 0 | \hat{a}_{k, 0}^\dagger | 0 \rangle u_k^*(x)) = 0$$

Zero Point Fluctuations

$$\langle 0 | \hat{\eta}(x)^2 | 0 \rangle$$

$$= \sum_k \frac{1}{4} L q_{0, k} q_{0, k} \langle 0 | \hat{a}_k^\dagger \hat{a}_k + 1 | 0 \rangle u_k(x) u_k^*(x)$$

$$= \sum_k \frac{1}{4} L q_{0, k}^2 |u_k(x)|^2 \neq 0$$

Thus $\Delta \eta(x) \neq 0$ with zero phonons in field

Note the famous divergence:

$$E_{vac} = \langle 0 | \hat{H} | 0 \rangle = \sum_k \frac{\hbar \omega_k}{2} \rightarrow \infty \text{ for } k \rightarrow \infty$$

Are our Phonons waves or particles?

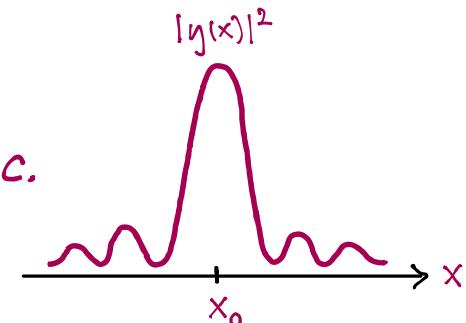
Extended

Localized

Particle-like Phonons

Classical Wavepacket

$$\eta(x) = \sum_k f_k u_k(x) + C.C.$$



Define $\hat{A}^+ = \sum_k f_k \hat{a}_k^\dagger$, $\sum_k |f_k|^2 = 1$

$\Rightarrow \hat{A}^+ |0\rangle = f_{k_1} |1_{k_1}, 0_{k_2}, \dots\rangle + f_{k_2} |0_{k_1}, 1_{k_2}, \dots\rangle + \dots$

Localized excitation in the field.

These Particle-like Phonons are Bosons

$$\hat{A}^{i+} \hat{A}^+ |0\rangle = \hat{A}^+ \hat{A}^{i+} |0\rangle$$

1st particle @ x
2nd particle @ x'

1st particle @ x'
2nd particle @ x