Begin 03-12-2024

Maxwell-Bloch Equations

So far in the Semiclassical Description

(*) Classical light acting on quantum atoms

(*) Next: Close the loop

Self-Consistent Description

Electric Field $\rightarrow$ 2-Level Atom

We need to set up and solve a set of workable simultaneous equations for the atoms and field.

(1) The <u>electric field</u>. We write

$$\boxed{\vec{E}(z,t) = \vec{\varepsilon}\,\underbrace{\mathcal{E}(z,t)}_{\text{wavepacket envelope}}\, e^{-i(\omega t - kz)}}$$

– Plane wave propagating in the $+z$ direction, the real part is the physical field

Slowly Varying Envelope Approximation (SVEA)

We require that the envelope $\mathcal{E}(z,t)$ is smooth in space and time compared to the plane wave part.

$$\left|\frac{\partial \mathcal{E}}{\partial z}\right| \ll k|\mathcal{E}| \qquad \left|\frac{\partial \mathcal{E}}{\partial t}\right| \ll \omega|\mathcal{E}|$$

$$\left|\frac{\partial^2 \mathcal{E}}{\partial z^2}\right| \ll k\left|\frac{\partial \mathcal{E}}{\partial z}\right| \qquad \left|\frac{\partial^2 \mathcal{E}}{\partial z^2}\right| \ll \omega\left|\frac{\partial \mathcal{E}}{\partial z}\right|$$

This is not particularly restrictive, unless working with ultrafast lasers.

(2) <u>The Macroscopic Polarization Density</u>.

We use the quantum expectation value $\vec{P}(z,t) = N\langle \hat{\vec{p}} \rangle$

Of this, we need the complex part that goes with $\vec{E}(z,t)$ and can be plugged into the wave equation.

Maxwell-Bloch Equations

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$$\left|\frac{\partial^2 \mathcal{E}}{\partial z^2}\right| \ll k\left|\frac{\partial \mathcal{E}}{\partial z}\right|, \qquad \left|\frac{\partial^2 \mathcal{E}}{\partial z^2}\right| \ll \omega\left|\frac{\partial \mathcal{E}}{\partial z}\right|$$

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(2) The Macroscopic Polarization Density.

We use the quantum expectation value $\vec{P}(z,t) = N\langle\hat{\vec{\mu}}\rangle$

Of this, we need the complex part that goes with $\vec{E}(z,t)$ and can be plugged into the wave equation.

Thus, of

$$\langle\hat{\vec{\mu}}\rangle = \vec{\mu}_{12}\langle a_2 a_1^*\rangle + \vec{\mu}_{21}\langle a_1 a_2^*\rangle$$
$$= \vec{\mu}_{12}\rho_{21}e^{-i(\omega t - kz)} + \vec{\mu}_{21}\rho_{12}e^{i(\omega t - kz)}$$

we need the part that goes as $e^{-i(\omega t - kz)}$

The *physical field* is $\mathrm{Re}[\hat{\varepsilon}\,\mathcal{E}(z,t)e^{-i(\omega t - kz)}]$

The *physical dipole* is

$$\mathrm{Re}\left[\boxed{\vec{\mu}_{12}\rho_{21}e^{-i(\omega t - kz)}} + \vec{\mu}_{21}\rho_{12}e^{i(\omega t - kz)}\right]$$

Note factor of 2 → $= \boxed{2\times}\,\mathrm{Re}[\vec{\mu}_{12}\rho_{21}e^{-i(\omega t - kz)}]$

Note: The coherence ρ_{21} depends on z,t because the field depends on z,t through the envelope $\mathcal{E}(z,t)$ ⇒ implicit SVEA on ρ_{21}.

Note: In a real, multilevel atom $\vec{\mu}_{12}$ need not be parallel to the field. However, only the part that is parallel to the field can emit radiation that interferes with it and lead to absorption and dispersion.

Maxwell-Bloch Equations

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slow variables

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Note: In a real, multilevel atom $\vec{\mu}_{12}$ need not be parallel to the field. However, only the part that is parallel to the field can emit radiation that interferes with it and lead to absorption and dispersion.

The complex dipole parallel to $\vec{\varepsilon}$ is

$$\vec{P}(z,t) = \vec{\varepsilon}\,2N(\vec{\mu}_{12}\cdot\vec{\varepsilon}^*)\,\rho_{21}(z,t)\,e^{-i(\omega t - kz)}$$

$$(|\varepsilon\rangle\langle\varepsilon|)|\mu\rangle \Rightarrow \vec{\mu}_{12}^{\parallel} = \begin{pmatrix}\varepsilon_x\\ \varepsilon_y\end{pmatrix}(\varepsilon_x^*, \varepsilon_y^*)\begin{pmatrix}\mu_{12}^{(x)}\\ \mu_{12}^{(y)}\end{pmatrix} = \vec{\varepsilon}\,(\vec{\mu}_{12}\cdot\vec{\varepsilon}^*)$$

Milloni & Eberly notation

Maxwell-Bloch Equations

Thus, of

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The complex dipole parallel to $\hat{\varepsilon}$ is

$$\vec{P}(z,t) = \hat{\varepsilon}\, 2N \underbrace{(\vec{\mu}_{12} \cdot \hat{\varepsilon}^*)}\, \rho_{21}(z,t)\, e^{-i(\omega t - kz)}$$

$$\equiv \hat{\varepsilon}\, 2N \mu^* \rho_{21}(z,t) e^{-i(\omega t - kz)}$$

<u>Final Note</u>: Because of the RWA we have

$$\left|\frac{\partial \rho_{21}}{\partial t}\right| \ll \omega |\rho_{21}| \qquad \left|\frac{\partial^2 \rho_{21}}{\partial t^2}\right| \ll \omega \left|\frac{\partial \rho_{21}}{\partial t}\right|$$

(3) Maxwells eqs. ➡ Wave Equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}(z,t) = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\vec{P}(z,t)$$

We plug in the complex $\vec{E}(z,t)$ and $\vec{P}(z,t)$, use the SVEA conditions on the derivatives to eliminate all but the leading terms, and finally take the scalar product with $\hat{\varepsilon}$ to get a scalar equation. **(Home Work Problem)**

Maxwell-Bloch Equations

The complex dipole parallel to $\vec{\mathcal{E}}$ is

$$\vec{P}(z,t) = \vec{\mathcal{E}}\, 2N \underbrace{(\vec{\mu}_{12}\cdot\vec{\mathcal{E}}^*)}\, \rho_{21}(z,t)\, e^{-i(\omega t - kz)}$$
$$\equiv \vec{\mathcal{E}}\, 2N \mu^* \rho_{21}(z,t)\, e^{-i(\omega t - kz)}$$

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This gives us our final equation for the envelope:

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\mathcal{E}(z,t) = \frac{ik}{\varepsilon_0} N \mu^* \rho_{21}(z,t)$$
where $\mu^* = \vec{\mu}_{12}\cdot\vec{\mathcal{E}}^*$

Write ρ_{21} in terms of the Bloch variables to get the

Maxwell-Bloch Equations

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\mathcal{E}(z,t) = \frac{ik}{2\varepsilon_0} N \mu^* (u - iv)$$
$$\dot{u} = -\beta u + Im[\chi]w + \Delta v$$
$$\dot{v} = -\beta v + \Delta u + Re[\chi]w$$
$$\dot{w} = -\frac{1}{T_1}(1+w) - Re[\chi]v - Im[\chi]u$$

Note: The Maxwell-Bloch Equations are a key result. They lead to rich physics, including absorption, gain, dispersion, solitons, lasers, and much more.

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\end{aligned}}$$

Note: The Maxwell-Bloch Equations are a key result. They lead to rich physics, including absorption, gain, dispersion, self-induced transparency, solitons, lasers, and much more.

Steady-State Solutions to MBE's

Steady state means that

$$\frac{1}{c}\frac{\partial\mathcal{E}}{\partial t}=0 \quad \& \quad \rho_{21}(z,t)\rightarrow\rho_{21}(z,\infty)=\frac{-i\chi/2}{\beta+i\Delta}(\rho_{22}-\rho_{11})$$

Combine with $\chi=-\vec{\mu}_{21}\cdot\hat{\varepsilon}\,\mathcal{E}/\hbar=\mu\mathcal{E}/\hbar$

$$\frac{\partial\mathcal{E}}{\partial z}=\frac{ik}{\varepsilon_0}N\mu^*\left(\frac{-i\mu\mathcal{E}}{2\hbar}\right)\frac{1}{\beta+i\Delta}(\rho_{22}-\rho_{11})$$

$$=\frac{kN|\mu|^2}{2\hbar\varepsilon_0}\frac{\beta-i\Delta}{\Delta^2+\beta^2}(\rho_{22}-\rho_{11})\mathcal{E}$$

We can rewrite this as

$$\boxed{\begin{aligned}
&\frac{\partial\mathcal{E}}{\partial z}=\frac{1}{2}(a-i\delta)w\mathcal{E}\\
&a=\frac{kN|\mu|^2}{2\hbar\varepsilon_0}\frac{\beta}{\Delta^2+\beta^2}\\
&\delta=\frac{kN|\mu|^2}{2\hbar\varepsilon_0}\frac{\Delta}{\Delta^2+\beta^2}=\frac{\Delta}{\beta}a
\end{aligned}}$$

Maxwell-Bloch Equations

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Combine with $\chi = -\vec{\mu}_{21}\cdot\vec{\varepsilon}\,\mathcal{E}/\hbar = \mu\mathcal{E}/\hbar$

$$\frac{\partial\mathcal{E}}{\partial z} = \frac{ik}{\varepsilon_0} N\mu^* \left(\frac{-i\mu\mathcal{E}}{2\hbar}\right)\frac{1}{\beta+i\Delta}(\rho_{22}-\rho_{11})$$

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We can rewrite this as

$$\boxed{\begin{aligned} \frac{\partial\mathcal{E}}{\partial z} &= \frac{1}{2}(a - i\delta)\,\omega\,\mathcal{E} \\ a &= \frac{kN|\mu|^2}{2\hbar\varepsilon_0}\frac{\beta}{\Delta^2+\beta^2} \\ \delta &= \frac{kN|\mu|^2}{2\hbar\varepsilon_0}\frac{\Delta}{\Delta^2+\beta^2} = \frac{\Delta}{\beta}\,a \end{aligned}}$$

To compare with our classical theory of dispersion, we solve for $\mathcal{E}(z)$ and plug into eq. for a plane wave.

Field: $E(z) = \mathcal{E}(z)e^{ikz}$

Envelope: $\mathcal{E}(z) = \mathcal{E}(0)\,e^{\left(\frac{a\omega}{2}\right)z}\,e^{i\left(-\frac{\delta\omega}{2}\right)z}$

Field: $E(z) = \mathcal{E}(0)\,e^{\left(\frac{a\omega}{2}\right)z}\,e^{i\left(1-\frac{\delta\omega}{2k}\right)kz}$

<u>Compare</u> to: $E(z) = E_0\,e^{-n_I kz}\,e^{i n_R kz}$

Real & Imaginary Index of Refraction

$$\boxed{\begin{aligned} n_I &= -\frac{a\omega}{2k} = -\frac{N\omega}{2k} \\ n_R &= 1 - \frac{\delta\omega}{2k} = 1 - \frac{\Delta}{\beta}\frac{N\omega}{2k} \end{aligned}}$$

Analogous to results from Electron Oscillator

$$n_I(\omega) = \frac{Ne^2}{4\varepsilon_0 m_e \omega} \Rightarrow \frac{\beta}{\Delta^2+\beta^2}, \quad n_R(\omega) = 1 + \frac{\Delta}{\beta}\,n_I(\omega)$$

Maxwell-Bloch Equations

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Analogous to results from Electron Oscillator

$$n_I(\omega) = \frac{Ne^2}{4\varepsilon_0 m_e \omega}\frac{\beta}{\Delta^2+\beta^2}\;,\quad n_R(\omega) = 1+\frac{\Delta}{\beta}n_I(\omega)$$

Behavior of the Intensity

$$\frac{\partial}{\partial z}|\mathcal{E}^*\mathcal{E}| = \mathcal{E}^*\frac{\partial\mathcal{E}}{\partial z} + \frac{\partial\mathcal{E}^*}{\partial z}\mathcal{E}$$

$$= \frac{1}{2}(a-id)\omega|\mathcal{E}|^2 + \frac{1}{2}(a+id)\omega|\mathcal{E}|^2 = a\omega|\mathcal{E}|^2$$

$$\boxed{\frac{\partial I}{\partial z} = a\omega I = a(\rho_{22}-\rho_{11})I}$$

Note that $\begin{cases} a \gtrless 0 \\ I(z) = I(0)e^{a(\rho_{22}-\rho_{11})z} \end{cases}$

Exp. Decay of I for	$\rho_{22}-\rho_{11} < 0$
Exp. growth of I for	$\rho_{22}-\rho_{11} > 0$

must be maintained by some external process

Maxwell-Bloch Equations

Behavior of the Intensity

$$\frac{\partial}{\partial z}|\mathcal{E}^*\mathcal{E}| = \mathcal{E}^*\frac{\partial \mathcal{E}}{\partial z} + \frac{\partial \mathcal{E}^*}{\partial z}\mathcal{E}$$

$$= \frac{1}{2}(a-id)w|\mathcal{E}|^2 + \frac{1}{2}(a+id)w|\mathcal{E}|^2 = aw|\mathcal{E}|^2$$

$$\boxed{\frac{\partial I}{\partial z} = awI = a(\rho_{22}-\rho_{11})I}$$

Note that $\begin{cases} a \geq 0 \\ I(z) = I(0)\,e^{a(\rho_{22}-\rho_{11})z} \end{cases}$

Exp. Decay of I for	$\rho_{22}-\rho_{11} < 0$
Exp. growth of I for	$\rho_{22}-\rho_{11} > 0$

must be maintained by some external process

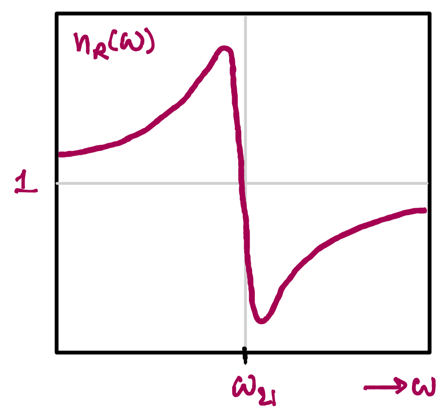
Behavior of the Dispersion:

Real & Imaginary Index of Refraction

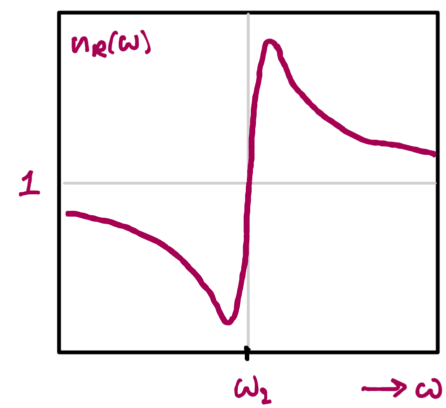
$$n_I = -\frac{aw}{2k} = -\frac{Nw}{2k}$$

$$n_R = 1 - \frac{\delta w}{2k} = 1 - \frac{\Delta}{\beta}\frac{Nw}{2k}$$

$w<1$ **absorption**



$w>1$ **gain**



End 03-12-2024

Not discussed in detail in class

Maxwell-Bloch Equations - Solitons

Self-Induced Transparency & Solitons

(*) Example of a non-trivial application of the MBE's in the context of pulse propagation (highly dynamic, non-steady state behavior.

(*) The pulse area theorem suggests a light pulse with the proper envelope will act as a 2π pulse. Thus, if the pulse is shorter than the excited state lifetime it may propagate without loss. Correct shaping may also allow propagation without changes in pulse shape.

(*) See Lecture Notes, Slusher & Gibbs 1972.

Envelope: $\mathcal{E}(z,t) = \frac{2\hbar}{\mu\tau}\,\mathrm{sech}(\xi/\tau),\ \xi = t - z/V,\ \Delta = 0$

$\Rightarrow\ \chi(z,t) = \frac{2}{\tau}\,\mathrm{sech}(\xi/\tau),\quad \theta = \int_{-\infty}^{\infty}\chi(\xi/\tau)\,dt = 2\pi$

Self-consistent solution with the the properties of a Soliton

$$\boxed{\begin{aligned}
&\mathcal{E}(z,t) = \frac{2\hbar}{\mu\tau}\,\mathrm{sech}(\xi/\tau)\\
&u(\xi/\tau) = 0\\
&v(\xi/\tau) = 2\,\mathrm{sech}(\xi/\tau)\tanh(\xi/\tau)\\
&w(\xi/\tau) = -1 + 2\,\mathrm{sech}(\xi/\tau)
\end{aligned}}$$

In the SVEA version of the Wave Eq.

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\mathcal{E}(z,t) = \frac{ik}{\varepsilon_0}N\mu^*(u - iv)$$

Substitute solutions for $\mathcal{E}$, u and v to get

$$\frac{2\hbar}{\mu\tau}\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial z}\right)\mathrm{sech}\left(\frac{t - zV}{\tau}\right) =$$

$$\frac{2\hbar}{\mu\tau}\left(\frac{-1}{V\tau} + \frac{1}{c\tau}\right)\left[-\mathrm{sech}\left(\frac{t-zV}{\tau}\right)\tanh\left(\frac{t-zV}{\tau}\right)\right] =$$

$$\frac{2kN\mu^*}{\varepsilon_0}\,\mathrm{sech}\left(\frac{t-zV}{\tau}\right)\tanh\left(\frac{t-zV}{\tau}\right)$$

Solve for c/V to get

$$\frac{c}{V} = 1 + \frac{kN|\mu|^2}{2\varepsilon_0\hbar}c\tau^2 = 1 + \frac{1}{2}a\beta c\tau^2$$

where $a = \frac{kN|\mu|^2}{2\varepsilon_0\hbar\beta} = N\sigma(0)$ (on-resonance absorption coeff.)

Consider Na vapor, $\lambda = 589\,nm$, $N = 10^{18}m^{-3}$, $T \sim 0K$, and $\beta = 2\pi \times 4.9\,MHz$ (completely opaque on res.)

Assuming $V \sim \frac{1}{2}c \Rightarrow \frac{c}{V} \sim 2 = 1 + a\beta c\tau^2 \Rightarrow \frac{1}{2}a\beta c\tau^2 \sim 1$

we must have $\tau \sim \sqrt{\frac{2}{a\beta c}} \sim 36\,ps \ll 16\,ns$!!