OPTI 544 Solution Set 4, Spring 2024

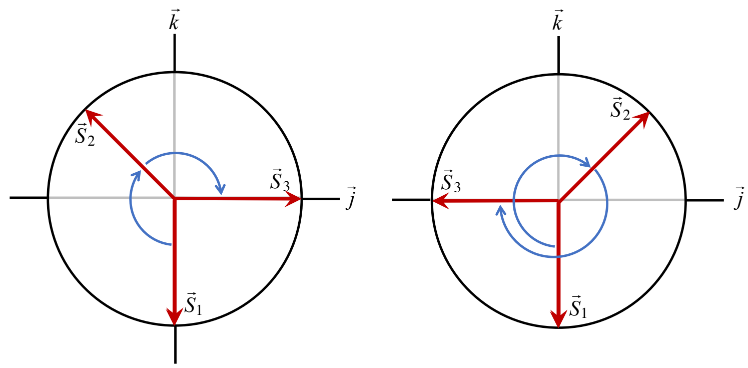
Problem 1

(a) Because χ is real and positive ($\varphi = 0$) we have a torque vector $\vec{Q} = -\chi\vec{i}$ and the Bloch vector $\vec{S}$ precesses in the $\vec{j} - \vec{k}$ plane. At $t = \pi/\chi_0$ the precession angles for the two atoms are $3\pi/4$ and $5\pi/4$, respectively.

<u>Bloch Vectors</u>

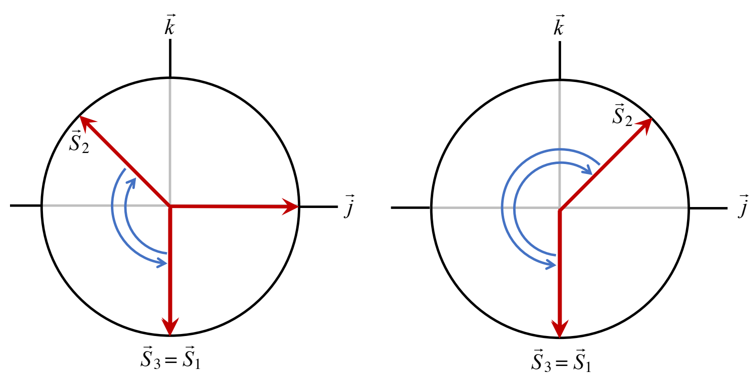
$\vec{S}_1$ at $t = 0$
$\vec{S}_2$ at $t = \pi/\chi_0$
$\vec{S}_3$ at $t = 2\pi/\chi_0$

Showing $\vec{j} - \vec{k}$ plane,
$\vec{i}$ points into the page



(b) Changing phase by 180º changes $\chi \to -\chi$ and thus the direction of precession, but the rate of precession is unchanged. The second pulse thus undoes the precession during the first pulse irrespective of the value of χ, and thus returns the Bloch vector to its initial state.

<u>Bloch Vectors</u>



Problem 2

Before we start, we adapt the shorthand notation $\frac{\partial}{\partial z}\mathcal{E}=\mathcal{E}'$, $\frac{\partial}{\partial t}\mathcal{E}=\dot{\mathcal{E}}$ and so on. ($\mathcal{E}$ is the best I can do for $\mathscr{E}$ in my equation editor)

(a) We start from the wave equation in a polarizable dielectric medium,

$$\left(\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(z,t)=\frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\mathbf{P}(z,t)$$

We plug in $\mathbf{E}(z,t)=\vec{\varepsilon}\mathcal{E}(z,t)e^{-i(\omega t-kz)}$ and $\mathbf{P}(z,t)=\vec{\varepsilon}\,2N\mu^*\rho_{21}(z,t)e^{-i(\omega t-kz)}$, take the scalar product with $\vec{\varepsilon}$ on both sides to remove the vector character, and expand out the derivatives. For the LHS we get

$$\mathcal{E}''e^{-i(\omega t-kz)}+2ik\mathcal{E}'e^{-i(\omega t-kz)}-k^2\mathcal{E}e^{-i(\omega t-kz)}-\frac{1}{c^2}\ddot{\mathcal{E}}e^{-i(\omega t-kz)}+\frac{i2\omega}{c^2}\dot{\mathcal{E}}e^{-i(\omega t-kz)}+\frac{\omega^2}{c^2}\mathcal{E}e^{-i(\omega t-kz)}$$

$$=2ik\left(\mathcal{E}'+\frac{1}{c^2}\dot{\mathcal{E}}\right)e^{-i(\omega t-kz)}$$

where, according to the SVEA, we have dropped the $\mathcal{E}''$ and $\ddot{\mathcal{E}}$ terms, and used $k^2-\omega^2/c^2=0$ to simplify. Note that the latter does not preclude a complex index of refraction, as the effect of absorption and dispersion will be incorporated into the complex envelope $\mathcal{E}(z,t)$.

On the RHS of the wave equation we have

$$\frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\mathbf{P}(z,t)=\frac{2N\mu^*}{\varepsilon_0 c^2}(\ddot{\rho}_{21}-i2\omega\dot{\rho}_{21})e^{-i(\omega t-kz)}=-\frac{2k^2N\mu^*}{\varepsilon_0}\rho_{21}(z,t)e^{-i(\omega t-kz)}.$$

where we have dropped the $\ddot{\rho}_{21}$ and $\dot{\rho}_{21}$. Putting everything together, dividing out the plane wave component, and rearranging, we finally get

$$\boxed{\left(\frac{\partial}{\partial z}-\frac{1}{c}\frac{\partial}{\partial t}\right)\mathcal{E}(z,t)=\frac{ik}{\varepsilon_0}N\mu^*\rho_{21}(z,t)}$$

(b) In steady state we substitute $\rho_{21}(\infty)=-i\frac{\chi}{2}\frac{\beta-i\Delta}{\Delta^2+\beta^2}(\rho_{22}-\rho_{11})$ (from Homework Set 3)

We also set $\dot{\mathcal{E}}=0$ and use $\chi=\vec{p}_{21}\cdot\vec{\varepsilon}\mathcal{E}/\hbar=\mu\mathcal{E}/\hbar$. This gives us

$$\frac{\partial}{\partial z}\mathcal{E} = \frac{kN|\mu|^2}{2\varepsilon_0}\frac{\beta - i\Delta}{\Delta^2+\beta^2}(\rho_{22}-\rho_{11})\mathcal{E} = \frac{1}{2}(a - i\delta)(\rho_{22}-\rho_{11})\mathcal{E}$$

where $$a = \frac{kN|\mu|^2}{\varepsilon_0\hbar}\frac{\beta}{\Delta^2+\beta^2}, \qquad \delta = \frac{kN|\mu|^2}{\varepsilon_0\hbar}\frac{\Delta}{\Delta^2+\beta^2}$$

These are the results from Class, from my handwritten notes, and from Milloni and Eberly.

Problem 3

(a) We have

$$\langle\hat{\vec{p}}\rangle = Tr\left[\hat{\vec{p}}\,\hat{\rho}\right] = Tr\left[\begin{pmatrix} 0 & \vec{p}_{12} \\ \vec{p}_{21} & 0 \end{pmatrix}\begin{pmatrix} \rho_{11} & \rho_{12}e^{i\omega t} \\ \rho_{21}e^{-i\omega t} & \rho_{22} \end{pmatrix}\right] = Tr\left[\begin{pmatrix} \vec{p}_{12}\rho_{21}e^{-i\omega t} & \vec{p}_{12}\rho_{22} \\ \vec{p}_{21}\rho_{11} & \vec{p}_{21}\rho_{12}e^{i\omega t} \end{pmatrix}\right]$$

$$= \vec{p}_{12}\rho_{21}e^{-i\omega t} + \vec{p}_{21}\rho_{12}e^{i\omega t} = \mathrm{Re}\left[2\vec{p}_{12}\rho_{21}e^{-i\omega t}\right]$$

In steady state we have $\rho_{21} = \dfrac{\chi}{2}\dfrac{\Delta + i\beta}{\Delta^2 + \beta^2 + |\chi|^2\beta/A_{21}}$ (from Problem 2-b above)

We plug this in above and get $\langle\hat{\vec{p}}\rangle = \mathrm{Re}\left[\vec{p}_{12}\dfrac{\beta + i\Delta}{\Delta^2 + \beta^2 + |\chi|^2\beta/A_{21}}\chi e^{-i\omega t}\right]$.

Now, if $A_{21} = 2\beta$ we can use $\dfrac{|\chi|^2}{A_{21}} = \dfrac{I}{I_{Sat}}$ to rewrite the above as

$$\boxed{\langle\hat{\vec{p}}\rangle = \mathrm{Re}\left[\vec{p}_{12}\frac{\Delta + i\beta}{\Delta^2 + \beta^2(1 + I/I_{Sat})}\chi e^{-i\omega t}\right]}$$

(b) Noting that $\chi = \vec{p}_{21}\cdot\vec{\varepsilon}\,\mathcal{E}/\hbar = \mu\mathcal{E}/\hbar$, where $\mu = \vec{p}_{21}\cdot\vec{\varepsilon}$, we have

$$\vec{p}'(t) = \vec{p}_{12}\frac{\Delta + i\beta}{\Delta^2 + \beta^2 + |\chi|^2\beta/A_{21}}\chi e^{-i\omega t} = \frac{\vec{p}_{12}\mu}{\hbar}\frac{\Delta + i\beta}{\Delta^2 + \beta^2 + |\chi|^2\beta/A_{21}}\mathcal{E}e^{-i\omega t}$$

The part of $\vec{p}'(t)$ parallel to $\mathbf{E}$ is (remember $\vec{\varepsilon}$ is complex)

$$\vec{p}(t) = \left(\vec{p}'(t)\cdot\vec{\varepsilon}^{\,*}\right)\vec{\varepsilon} = \frac{|\mu|^2}{\hbar}\frac{\Delta + i\beta}{\Delta^2 + \beta^2 + |\chi|^2\beta/A_{21}}\vec{\varepsilon}\mathcal{E}e^{-i\omega t} \equiv \alpha(\omega)\mathbf{E}(t)$$

We thus obtain

$$\boxed{\alpha(\omega) = \frac{|\mu|^2}{\hbar}\frac{\Delta + i\beta}{\Delta^2 + \beta^2 + |\chi|^2\beta/A_{21}} = \frac{|\mu|^2}{\hbar}\frac{\Delta + i\beta}{\Delta^2 + \beta^2(1 + I/I_{Sat})}}$$

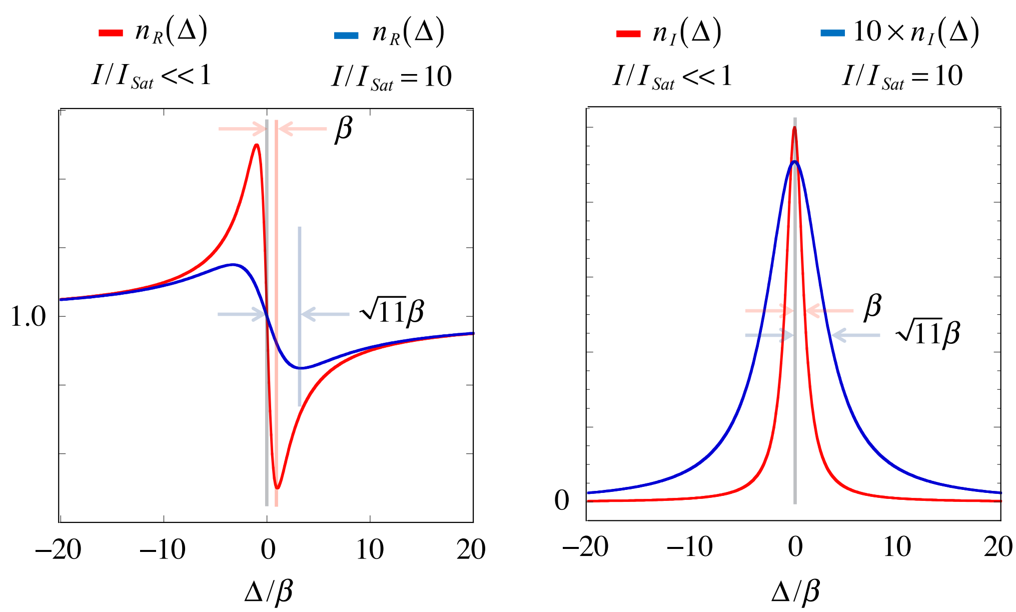
where the second step is true when $A_{21} = 2\beta$. We will assume this is the case in part (c).

(c) For $n(\omega)\approx 1$ we have

$$n_R(\omega) = 1+\frac{N}{2\varepsilon_0}\mathrm{Re}[\alpha(\omega)] = 1+\frac{N|\mu|^2}{2\varepsilon_0\hbar}\frac{\Delta}{\Delta^2+\beta^2(1+I/I_{Sat})}$$

$$n_I(\omega) = \frac{N}{2\varepsilon_0}\mathrm{Im}[\alpha(\omega)] = \frac{N|\mu|^2}{2\varepsilon_0\hbar}\frac{\beta}{\Delta^2+\beta^2(1+I/I_{Sat})}$$

Sketch for $I/I_{Sat} << 1$ and for $I/I_{Sat} = 10$:



(d) For $I/I_{Sat} >> 1$ both the dispersion and absorption features are broadened. In the sketch we have set $I/I_{Sat} = 10$, which makes the power broadened linewidth

$$\beta' = \beta\sqrt{1+I/I_{Sat}} = \sqrt{11}\beta \sim 3.3\beta$$

At the same time, the peak dispersion is reduced by a factor

$$1/\sqrt{1+I/I_{Sat}} = 1/\sqrt{11} = 0.30$$

And the peak absorption is reduced by a factor

$$1/(1+I/I_{Sat}) = 1/11 = 0.091.$$