## OPTI 544 Solution Set 3, Spring 2024

## Problem 1

(a) The equations of motion for the probability amplitudes in the RWA, using the "slow" variables and setting $\Delta=\delta=0$, are

$$
\dot{b}_{1}=-i \frac{\chi_{1}}{2} b_{2}, \quad \dot{b}_{2}=-i\left(\frac{\chi_{1}}{2} b_{1}+\frac{\chi_{2}}{2} b_{3}\right), \quad \dot{b}_{3}=-i \frac{\chi_{2}}{2} b_{2}
$$

(b) If $b_{2}(0)=0$ then $b_{2}(t)=0$ if and only if $\dot{b}_{2}=0$ at all times. That requires

$$
\left(\frac{\chi_{1}}{2} b_{1}+\frac{\chi_{2}}{2} b_{3}\right)=0 \Rightarrow b_{3}=-\frac{\chi_{1}}{\chi_{2}} b_{1}
$$

Now

$$
\left|b_{1}\right|^{2}+\left|b_{3}\right|^{2}=1 \Rightarrow\left|b_{1}\right|^{2}+\frac{\chi_{1}^{2}}{\chi_{2}^{2}}\left|b_{1}\right|^{2}=1 \Rightarrow\left|b_{1}\right|^{2}=\frac{\chi_{1}^{2}}{\chi_{1}^{2}+\chi_{2}^{2}} .
$$

Consistent with the above, we choose

$$
b_{1}=\frac{\chi_{2}}{\sqrt{\chi_{1}^{2}+\chi_{2}^{2}}}, \quad b_{3}=\frac{-\chi_{1}}{\sqrt{\chi_{1}^{2}+\chi_{2}^{2}}}
$$

Note: If $b_{2}(t)=0$ then $\dot{b}_{1}=\dot{b}_{3} \Rightarrow b_{1}, b_{3}$ are constant
(c) If the driving field does not lead to a non-zero probability amplitude in the excited state $|2\rangle$, then there can be no induced dipole moment. This is because $|1\rangle$ and $|3\rangle$ of necessity must be of the same parity to allow for Raman coupling in the first place. That means there can be no absorption or emission of light, and as a result the wave propagates without loss of intensity.

Note: The state found in (b) above is referred to as a "dark state".

## Problem 2

(a) Taking the outer product of the state vectors we find
$\rho_{1}=\frac{1}{3}\left(\begin{array}{c}1 \\ i \sqrt{2} \\ 0\end{array}\right)(1,-i \sqrt{2}, 0)=\frac{1}{3}\left(\begin{array}{ccc}1 & -i \sqrt{2} & 0 \\ i \sqrt{2} & 2 & 0 \\ 0 & 0 & 0\end{array}\right) . \quad$ Check: $\operatorname{Tr}\left[\rho_{1}\right]=1$
$\rho_{2}=\frac{1}{5}\left(\begin{array}{c}1+i \\ 1 \\ -i \sqrt{2}\end{array}\right)(1-i, i \sqrt{2})=\frac{1}{5}\left(\begin{array}{ccc}2 & 1+i & -(1-i) \\ 1-i & 1 & i \sqrt{2} \\ -(1+i) & -i \sqrt{2} & 2\end{array}\right) . \quad$ Check: $\operatorname{Tr}\left[\rho_{2}\right]=1$

Then $\quad \rho=\frac{1}{2}\left(\rho_{1}+\rho_{2}\right)=\frac{1}{2} \times \frac{1}{15}\left[\left(\begin{array}{ccc}1 & -i \sqrt{2} & 0 \\ i \sqrt{2} & 2 & 0 \\ 0 & 0 & 0\end{array}\right)+\left(\begin{array}{ccc}2 & 1+i & -(1-i) \\ 1-i & 1 & i \sqrt{2} \\ -(1+i) & -i \sqrt{2} & 2\end{array}\right)\right]$

$$
=\frac{1}{30}\left(\begin{array}{ccc}
11 & 3+i(3-5 \sqrt{2}) & -(1-i) 3 \sqrt{2} \\
3-i(3-5 \sqrt{2)} & 13 & i 3 \sqrt{2} \\
-(1+i) 3 \sqrt{2} & -i 3 \sqrt{2} & 6
\end{array}\right) . \quad \text { Check: } \operatorname{Tr}[\rho]=1
$$

(b) To check for purity we can compute $\rho^{2}$ and then check if $\rho^{2} \neq \rho$ or $\operatorname{Tr}\left[\rho^{2}\right]<1$, either of which would tell us that the state is mixed. For example, in this case it is straightforward though somewhat tedious to show that $\operatorname{Tr}\left[\rho^{2}\right]=(19-2 \sqrt{2}) / 30=0.539<1$, which tells us the state is mixed.

There is an easier way if we know the ensemble decomposition, $\rho=\Sigma_{k} p_{k}\left|\psi_{k}\right\rangle\left\langle\psi_{k}\right|$. Namely, that $\rho$ is mixed if one or more of the $\left|\psi_{k}\right\rangle$ are linearly independent of the other. In our case we can confirm this by inspection, since $\left|\psi_{1}\right\rangle$ is confined to the $\left.\{1\rangle,|2\rangle\right\}$ subspace while $\left|\psi_{2}\right\rangle$ has a component along $|3\rangle$.

## Problem 3

(a) The Hamiltonian for our 2-level system has the form

$$
H=\hbar\left(\begin{array}{cc}
0 & -\frac{1}{2}\left(\chi_{12} e^{-i \omega t}+\chi_{21}^{*} e^{i \omega t}\right) \\
-\frac{1}{2}\left(\chi_{21} e^{-i \omega t}+\chi_{12}^{*} e^{i \omega t}\right) & \omega_{21}
\end{array}\right)
$$

Starting from $i \hbar \dot{\rho}=[H, \rho]$ we get $i \hbar \dot{\rho}_{i j}=\sum_{k}\left(H_{i k} \rho_{k j}-\rho_{i k} H_{k j}\right), i, j, k \in\{1,2\}$.
Applying this to the elements of the 2-level density matrix, we get

$$
\begin{aligned}
\dot{\rho}_{11}= & \frac{1}{i \hbar}\left(H_{11} \rho_{11}-\rho_{11} H_{11}+H_{12} \rho_{21}-\rho_{12} H_{21}\right) \\
& =\frac{i}{2}\left(\chi_{12} e^{-i \omega t}+\chi_{21}^{*} e^{i \omega t}\right) \rho_{21}-\frac{i}{2}\left(\chi_{21} e^{-i \omega t}+\chi_{12}^{*} e^{i \omega t}\right) \rho_{12}
\end{aligned}
$$

and

$$
\begin{aligned}
i \hbar \dot{\rho}_{12} & =H_{11} \rho_{12}-\rho_{11} H_{12}+H_{12} \rho_{22}-\rho_{12} H_{22} \\
& =i \omega_{21} \rho_{12}+\frac{i}{2}\left(\chi_{12} e^{-i \omega t}+\chi_{21}^{*} e^{i \omega t}\right)\left(\rho_{22}-\rho_{11}\right)
\end{aligned}
$$

Now let $\rho_{12}=\tilde{\rho}_{12} e^{i \omega t}, \rho_{21}=\tilde{\rho}_{21} e^{-i \omega t}$ and substitute in the above equations. This gives us

$$
\begin{aligned}
& \dot{\rho}_{11}=\frac{i}{2}\left(\chi_{12} e^{-i \omega t}+\chi_{21}^{*} e^{i \omega t}\right) \rho_{21}-\frac{i}{2}\left(\chi_{21} e^{-i \omega t}+\chi_{12}^{*} e^{i \omega t}\right) \rho_{12} \\
\Rightarrow \quad & \dot{\rho}_{11}=\frac{i}{2}\left(\chi_{12} e^{-2 i \omega t}+\chi_{21}^{*}\right) \tilde{\rho}_{21}-\frac{i}{2}\left(\chi_{21}+\chi_{12}^{*} e^{2 i \omega t}\right) \tilde{\rho}_{12} \\
& \dot{\rho}_{12}=\frac{d}{d t}\left(\tilde{\rho}_{12} e^{i \omega t}\right)=\dot{\tilde{\rho}}_{12} e^{i \omega t}+i \omega \tilde{\rho}_{12} e^{i \omega t}=i \omega_{21} \tilde{\rho}_{12} e^{i \omega t}+\frac{i}{2}\left(\chi_{12} e^{-i \omega t}+\chi_{21}^{*} e^{i \omega t}\right)\left(\rho_{22}-\rho_{11}\right) \\
\Rightarrow \quad & \dot{\tilde{\rho}}_{12}=i\left(\omega_{21}-\omega\right) \tilde{\rho}_{12} \frac{i}{2}\left(\chi_{12} e^{-i 2 \omega t}+\chi_{21}^{*}\right)\left(\rho_{22}-\rho_{11}\right)
\end{aligned}
$$

We set $\Delta=\left(\omega_{21}-\omega\right), \chi_{12}=\chi, \chi_{21}^{*}=\chi^{*}$, drop the terms $\propto e^{ \pm 2 \omega t}$, and use $\dot{\rho}_{22}=-\dot{\rho}_{11}$, $\dot{\rho}_{12}=\dot{\rho}_{21}^{*}$. This gives us the desired result

$$
\begin{array}{ll}
\dot{\rho}_{11}=-\frac{i}{2}\left(\chi \rho_{12}-\chi^{*} \rho_{21}\right), & \dot{\rho}_{22}=\frac{i}{2}\left(\chi \rho_{12}-\chi^{*} \rho_{21}\right) \\
\dot{\rho}_{12}=i \Delta \rho_{12}+\frac{i \chi^{*}}{2}\left(\rho_{22}-\rho_{11}\right), & \dot{\rho}_{21}=-i \Delta \rho_{21}-\frac{i \chi^{2}}{2}\left(\rho_{22}-\rho_{11}\right)
\end{array}
$$

(b) The Density Matrix has 2 real-valued populations and 2 complex-valued coherences, which suggests a total of 6 real-valued variables that must be known in order to specify $\rho$. However, the constraints $\rho_{22}=1-\rho_{11}$ and $\rho_{12}=\rho_{21}^{*}$ allow us to express 3 of the 6 variables in terms of the other 3. This leaves us with a total of 3 real-valued variables necessary to specify an arbitrary Density Matrix, whether it is pure or mixed.
(c) Major approximations implicit in the above result:
(i) The Electric Dipole Approximation (inherent in the form of $H$ )
(ii) The 2-Level Approximation
(iii) The Rotating Wave Approximation

## Problem 4

In steady state the Density Matrix Equations reduce to

$$
\begin{equation*}
\dot{\rho}_{11}=A_{21} \rho_{22}-\frac{i}{2}\left(\chi \rho_{12}-\chi^{*} \rho_{21}\right)=0 \tag{i}
\end{equation*}
$$

(ii) $\quad \dot{\rho}_{22}=-A_{21} \rho_{22}+\frac{i}{2}\left(\chi \rho_{12}-\chi^{*} \rho_{21}\right)=0$
(iii) $\quad \dot{\rho}_{12}=-(\beta-i \Delta) \rho_{12}+i \frac{\chi^{*}}{2}\left(\rho_{22}-\rho_{11}\right)=\dot{\rho}_{21}^{*}=0$

We start by solving for the coherences in Equation (iii)

$$
\rho_{12}=\frac{i \frac{\chi^{*}}{2}\left(\rho_{22}-\rho_{11}\right)}{\beta-i \Delta}=i \frac{\chi^{*}}{2} \frac{\beta+i \Delta}{\Delta^{2}+\beta^{2}}\left(\rho_{22}-\rho_{11}\right), \quad \rho_{21}=-i \frac{\chi^{*}}{2} \frac{\beta-i \Delta}{\Delta^{2}+\beta^{2}}\left(\rho_{22}-\rho_{11}\right)
$$

From this we get $\quad \chi \rho_{12}-\chi^{*} \rho_{21}=\frac{i|\chi|^{2} \beta}{\Delta^{2}+\beta^{2}}\left(\rho_{22}-\rho_{11}\right)$

Substituting in Equation (ii), using $\rho_{22}-\rho_{11}=2 \rho_{22}-1$, and solving for $\rho_{22}$, we get

$$
\begin{aligned}
& \rho_{22}=\frac{i}{2 A_{21}} \frac{i|\chi|^{2} \beta}{\Delta^{2}+\beta^{2}}\left(2 \rho_{22}-1\right) \Rightarrow\left(1+\frac{|\chi|^{2} \beta / A_{21}}{\Delta^{2}+\beta^{2}}\right) \rho_{22}=\frac{1}{2} \frac{|\chi|^{2} \beta / A_{21}}{\Delta^{2}+\beta^{2}} \\
& \Rightarrow \quad \rho_{22}=\frac{|\chi|^{2} \beta / 2 A_{21}}{\Delta^{2}+\beta^{2}+|\chi|^{2} \beta / A_{21}}
\end{aligned}
$$

Next,

$$
\rho_{11}=1-\rho_{22}=1-\frac{|\chi|^{2} \beta / 2 A_{21}}{\Delta^{2}+\beta^{2}+|\chi|^{2} \beta / A_{21}}=\frac{\Delta^{2}+\beta^{2}+|\chi|^{2} \beta / 2 A_{21}}{\Delta^{2}+\beta^{2}+|\chi|^{2} \beta / A_{21}}
$$

Sanity check: $\rho_{11}+\rho_{22}=1$. With that we have the steady state solutions

$$
\begin{array}{ll}
\rho_{11}(\infty)=\frac{\Delta^{2}+\beta^{2}+|\chi|^{2} \beta / 2 A_{21}}{\Delta^{2}+\beta^{2}+|\chi|^{2} \beta / A_{21}} & \rho_{22}(\infty)=\frac{|\chi|^{2} \beta / 2 A_{21}}{\Delta^{2}+\beta^{2}+|\chi|^{2} \beta / A_{21}} \\
\rho_{12}(\infty)=i \frac{\chi^{*}}{2} \frac{\beta+i \Delta}{\Delta^{2}+\beta^{2}}\left(\rho_{22}-\rho_{11}\right) & \rho_{21}(\infty)=-i \frac{\chi}{2} \frac{\beta-i \Delta}{\Delta^{2}+\beta^{2}}\left(\rho_{22}-\rho_{11}\right)
\end{array}
$$

