# **OPTI 544 Solution Set 3, Spring 2024**

### **Problem 1**

(a) The equations of motion for the probability amplitudes in the RWA, using the "slow" variables and setting  $\Delta = \delta = 0$ , are

$$\dot{b}_1 = -i\frac{\chi_1}{2}b_2, \qquad \dot{b}_2 = -i\left(\frac{\chi_1}{2}b_1 + \frac{\chi_2}{2}b_3\right), \qquad \dot{b}_3 = -i\frac{\chi_2}{2}b_2.$$

(b) If  $b_2(0) = 0$  then  $b_2(t) = 0$  if and only if  $\dot{b}_2 = 0$  at all times. That requires

$$\left(\frac{\chi_1}{2}b_1 + \frac{\chi_2}{2}b_3\right) = 0 \implies b_3 = -\frac{\chi_1}{\chi_2}b_1.$$

Now

$$|b_1|^2 + |b_3|^2 = 1 \implies |b_1|^2 + \frac{\chi_1^2}{\chi_2^2} |b_1|^2 = 1 \implies |b_1|^2 = \frac{\chi_1^2}{\chi_1^2 + \chi_2^2}.$$

Consistent with the above, we choose

$$b_1 = \frac{\chi_2}{\sqrt{\chi_1^2 + \chi_2^2}}$$
,  $b_3 = \frac{-\chi_1}{\sqrt{\chi_1^2 + \chi_2^2}}$ 

Note: If  $b_2(t) = 0$  then  $\dot{b}_1 = \dot{b}_3 \Rightarrow b_1$ ,  $b_3$  are constant

(c) If the driving field does not lead to a non-zero probability amplitude in the excited state |2⟩, then there can be no induced dipole moment. This is because |1⟩ and |3⟩ of necessity must be of the same parity to allow for Raman coupling in the first place. That means there can be no absorption or emission of light, and as a result the wave propagates without loss of intensity.

Note: The state found in (b) above is referred to as a "dark state".

#### Problem 2

(a) Taking the outer product of the state vectors we find

$$\rho_{1} = \frac{1}{3} \begin{pmatrix} 1 \\ i\sqrt{2} \\ 0 \end{pmatrix} (1, -i\sqrt{2}, 0) = \frac{1}{3} \begin{pmatrix} 1 & -i\sqrt{2} & 0 \\ i\sqrt{2} & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
 Check:  $\operatorname{Tr}[\rho_{1}] = 1$ 

$$\rho_{2} = \frac{1}{5} \begin{pmatrix} 1+i \\ 1 \\ -i\sqrt{2} \end{pmatrix} \left(1-i, i\sqrt{2}\right) = \frac{1}{5} \begin{pmatrix} 2 & 1+i & -(1-i) \\ 1-i & 1 & i\sqrt{2} \\ -(1+i) & -i\sqrt{2} & 2 \end{pmatrix}.$$
 Check:  $\operatorname{Tr}[\rho_{2}] = 1$ 

Then 
$$\rho = \frac{1}{2}(\rho_1 + \rho_2) = \frac{1}{2} \times \frac{1}{15} \left[ \begin{pmatrix} 1 & -i\sqrt{2} & 0 \\ i\sqrt{2} & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1+i & -(1-i) \\ 1-i & 1 & i\sqrt{2} \\ -(1+i) & -i\sqrt{2} & 2 \end{pmatrix} \right]$$

$$= \frac{1}{30} \begin{pmatrix} 11 & 3+i(3-5\sqrt{2}) - (1-i)3\sqrt{2} \\ 3-i(3-5\sqrt{2}) & 13 & i3\sqrt{2} \\ -(1+i)3\sqrt{2} & -i3\sqrt{2} & 6 \end{pmatrix}.$$
 Check:  $\operatorname{Tr}[\rho] = 1$ 

(b) To check for purity we can compute  $\rho^2$  and then check if  $\rho^2 \neq \rho$  or  $\text{Tr}[\rho^2] < 1$ , either of which would tell us that the state is mixed. For example, in this case it is straightforward though somewhat tedious to show that  $\text{Tr}[\rho^2] = (19 - 2\sqrt{2})/30 = 0.539 < 1$ , which tells us the state is mixed.

There is an easier way if we know the ensemble decomposition,  $\rho = \sum_k p_k |\psi_k\rangle \langle \psi_k|$ . Namely, that  $\rho$  is mixed if one or more of the  $|\psi_k\rangle$  are linearly independent of the other. In our case we can confirm this by inspection, since  $|\psi_1\rangle$  is confined to the  $\{1\rangle, |2\rangle\}$  subspace while  $|\psi_2\rangle$  has a component along  $|3\rangle$ .

#### **Problem 3**

(a) The Hamiltonian for our 2-level system has the form

$$H = \hbar \begin{pmatrix} 0 & -\frac{1}{2} (\chi_{12} e^{-i\omega t} + \chi_{21}^* e^{i\omega t}) \\ -\frac{1}{2} (\chi_{21} e^{-i\omega t} + \chi_{12}^* e^{i\omega t}) & \omega_{21} \end{pmatrix}$$

Starting from  $i\hbar\dot{\rho} = [H,\rho]$  we get  $i\hbar\dot{\rho}_{ij} = \sum_{k} (H_{ik}\rho_{kj} - \rho_{ik}H_{kj})$ ,  $i, j, k \in \{1,2\}$ . Applying this to the elements of the 2-level density matrix, we get

$$\dot{\rho}_{11} = \frac{1}{i\hbar} (H_{11}\rho_{11} - \rho_{11}H_{11} + H_{12}\rho_{21} - \rho_{12}H_{21})$$
  
=  $\frac{i}{2} (\chi_{12}e^{-i\omega t} + \chi_{21}^{*}e^{i\omega t})\rho_{21} - \frac{i}{2} (\chi_{21}e^{-i\omega t} + \chi_{12}^{*}e^{i\omega t})\rho_{12}$ 

and

$$i\hbar\dot{\rho}_{12} = H_{11}\rho_{12} - \rho_{11}H_{12} + H_{12}\rho_{22} - \rho_{12}H_{22}$$
$$= i\omega_{21}\rho_{12} + \frac{i}{2}(\chi_{12}e^{-i\omega t} + \chi_{21}^*e^{i\omega t})(\rho_{22} - \rho_{11})$$

Now let  $\rho_{12} = \tilde{\rho}_{12} e^{i\omega t}$ ,  $\rho_{21} = \tilde{\rho}_{21} e^{-i\omega t}$  and substitute in the above equations. This gives us

$$\begin{split} \dot{\rho}_{11} &= \frac{i}{2} \left( \chi_{12} e^{-i\omega t} + \chi_{21}^{*} e^{i\omega t} \right) \rho_{21} - \frac{i}{2} \left( \chi_{21} e^{-i\omega t} + \chi_{12}^{*} e^{i\omega t} \right) \rho_{12} \\ \Rightarrow & \dot{\rho}_{11} = \frac{i}{2} \left( \chi_{12} e^{-2i\omega t} + \chi_{21}^{*} \right) \tilde{\rho}_{21} - \frac{i}{2} \left( \chi_{21} + \chi_{12}^{*} e^{2i\omega t} \right) \tilde{\rho}_{12} \\ & \dot{\rho}_{12} = \frac{d}{dt} \left( \tilde{\rho}_{12} e^{i\omega t} \right) = \dot{\tilde{\rho}}_{12} e^{i\omega t} + i\omega \tilde{\rho}_{12} e^{i\omega t} = i\omega_{21} \tilde{\rho}_{12} e^{i\omega t} + \frac{i}{2} \left( \chi_{12} e^{-i\omega t} + \chi_{21}^{*} e^{i\omega t} \right) \left( \rho_{22} - \rho_{11} \right) \\ \Rightarrow & \dot{\tilde{\rho}}_{12} = i \left( \omega_{21} - \omega \right) \tilde{\rho}_{12} \frac{i}{2} \left( \chi_{12} e^{-i2\omega t} + \chi_{21}^{*} \right) \left( \rho_{22} - \rho_{11} \right) \end{split}$$

We set  $\Delta = (\omega_{21} - \omega)$ ,  $\chi_{12} = \chi$ ,  $\chi_{21}^* = \chi^*$ , drop the terms  $\propto e^{\pm i 2\omega t}$ , and use  $\dot{\rho}_{22} = -\dot{\rho}_{11}$ ,  $\dot{\rho}_{12} = \dot{\rho}_{21}^*$ . This gives us the desired result

$$\dot{\rho}_{11} = -\frac{i}{2} (\chi \rho_{12} - \chi^* \rho_{21}), \qquad \dot{\rho}_{22} = \frac{i}{2} (\chi \rho_{12} - \chi^* \rho_{21})$$
  
$$\dot{\rho}_{12} = i\Delta \rho_{12} + \frac{i\chi^*}{2} (\rho_{22} - \rho_{11}), \qquad \dot{\rho}_{21} = -i\Delta \rho_{21} - \frac{i\chi^2}{2} (\rho_{22} - \rho_{11})$$

- (b) The Density Matrix has 2 real-valued populations and 2 complex-valued coherences, which suggests a total of 6 real-valued variables that must be known in order to specify  $\rho$ . However, the constraints  $\rho_{22} = 1 \rho_{11}$  and  $\rho_{12} = \rho_{21}^*$  allow us to express 3 of the 6 variables in terms of the other 3. This leaves us with a total of 3 real-valued variables necessary to specify an arbitrary Density Matrix, whether it is pure or mixed.
- (c) Major approximations implicit in the above result:
  - (i) The Electric Dipole Approximation (inherent in the form of *H*)
  - (ii) The 2-Level Approximation
  - (iii) The Rotating Wave Approximation

## Problem 4

In steady state the Density Matrix Equations reduce to

(i) 
$$\dot{\rho}_{11} = A_{21}\rho_{22} - \frac{i}{2}(\chi\rho_{12} - \chi^*\rho_{21}) = 0$$

(ii) 
$$\dot{\rho}_{22} = -A_{21}\rho_{22} + \frac{i}{2}(\chi\rho_{12} - \chi^*\rho_{21}) = 0$$

(iii) 
$$\dot{\rho}_{12} = -(\beta - i\Delta)\rho_{12} + i\frac{\chi^*}{2}(\rho_{22} - \rho_{11}) = \dot{\rho}_{21}^* = 0$$

We start by solving for the coherences in Equation (iii)

$$\rho_{12} = \frac{i\frac{\chi^{*}}{2}(\rho_{22} - \rho_{11})}{\beta - i\Delta} = i\frac{\chi^{*}}{2}\frac{\beta + i\Delta}{\Delta^{2} + \beta^{2}}(\rho_{22} - \rho_{11}), \qquad \rho_{21} = -i\frac{\chi^{*}}{2}\frac{\beta - i\Delta}{\Delta^{2} + \beta^{2}}(\rho_{22} - \rho_{11})$$

From this we get  $\chi \rho_{12} - \chi^* \rho_{21} = \frac{i|\chi|^2 \beta}{\Delta^2 + \beta^2} (\rho_{22} - \rho_{11})$ 

Substituting in Equation (ii), using  $\rho_{22} - \rho_{11} = 2\rho_{22} - 1$ , and solving for  $\rho_{22}$ , we get

$$\rho_{22} = \frac{i}{2A_{21}} \frac{i|\chi|^2 \beta}{\Delta^2 + \beta^2} (2\rho_{22} - 1) \implies \left( 1 + \frac{|\chi|^2 \beta / A_{21}}{\Delta^2 + \beta^2} \right) \rho_{22} = \frac{1}{2} \frac{|\chi|^2 \beta / A_{21}}{\Delta^2 + \beta^2}$$
$$\implies \rho_{22} = \frac{|\chi|^2 \beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2 \beta / A_{21}}$$

Next,

$$\rho_{11} = 1 - \rho_{22} = 1 - \frac{|\chi|^2 \beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2 \beta / A_{21}} = \frac{\Delta^2 + \beta^2 + |\chi|^2 \beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2 \beta / A_{21}}$$

Sanity check:  $\rho_{11} + \rho_{22} = 1$ . With that we have the steady state solutions

$$\rho_{11}(\infty) = \frac{\Delta^{2} + \beta^{2} + |\chi|^{2} \beta / 2A_{21}}{\Delta^{2} + \beta^{2} + |\chi|^{2} \beta / A_{21}} \qquad \rho_{22}(\infty) = \frac{|\chi|^{2} \beta / 2A_{21}}{\Delta^{2} + \beta^{2} + |\chi|^{2} \beta / A_{21}}$$

$$\rho_{12}(\infty) = i\frac{\chi^{*}}{2}\frac{\beta + i\Delta}{\Delta^{2} + \beta^{2}}(\rho_{22} - \rho_{11}) \qquad \rho_{21}(\infty) = -i\frac{\chi}{2}\frac{\beta - i\Delta}{\Delta^{2} + \beta^{2}}(\rho_{22} - \rho_{11})$$