OPTI 544 Solution Set 2, Spring 2024

Problem I

(a) This is a straightforward exercise in solving a homogeneous 2nd order differential equation with known boundary conditions. It is a bit tedious, but a useful thing of which to remind ourselves since it will be a recurring step in many parts of the course.

We start from (i) $\dot{c}_1 = \frac{i\chi^*}{2} c_2$, (ii) $\dot{c}_2 = -i\Delta c_2 + \frac{i\chi}{2} c_1$

Take the time derivative of (ii) and substitute $\dot{c}_1$ from (i), to get

$$\ddot{c}_2 = -i\Delta\dot{c}_2 + \frac{i\chi}{2}\dot{c}_1 = -i\Delta\dot{c}_2 - \frac{|\chi|^2}{4}c_2 \quad\Rightarrow\quad \ddot{c}_2 + i\Delta\dot{c}_2 + \frac{|\chi|^2}{4}c_2 = 0$$

This is a homogeneous 2nd order differential equation that can be solved using standard methods.

First, the Characteristic Equation is $r^2 + i\Delta r + \frac{|\chi|^2}{4} = 0$, with roots

$$\lambda_\pm = -\frac{i\Delta}{2} \pm \frac{1}{2}\sqrt{-\Delta^2 - |\chi|^2} = -\frac{i}{2}(\Delta \pm \Omega)$$

The complete solution is therefore of the form

$$c_2(t) = \left(\alpha e^{-i\frac{\Omega t}{2}} + \beta e^{i\frac{\Omega t}{2}}\right)e^{-i\frac{\Delta t}{2}} = \left[\alpha\left(\cos\frac{\Omega t}{2} - i\sin\frac{\Omega t}{2}\right) + \beta\left(\cos\frac{\Omega t}{2} + i\sin\frac{\Omega t}{2}\right)\right]e^{-i\frac{\Delta t}{2}}$$
$$= \left[(a+\beta)\cos\frac{\Omega t}{2} - i(\alpha-\beta)\sin\frac{\Omega t}{2}\right]e^{-i\frac{\Delta t}{2}}$$

From the condition $c_2(0) = 1$ we see that $\alpha + \beta = 1$. To determine $\alpha - \beta = 1$ we examine $\dot{c}_2(0)$. First,

$$\dot{c}_2(t) = -\frac{i\alpha}{2}(\Delta+\Omega)e^{-\frac{i}{2}(\Delta+\Omega)t} - \frac{i\beta}{2}(\Delta-\Omega)e^{-\frac{i}{2}(\Delta-\Omega)t} = -i\Delta c_2(t) + \frac{i\chi}{2}c_1(t)$$

At $t = 0$ we have $c_2(0) = 1$, $c_1(0) = 0$, so we get

$$-\frac{i\alpha}{2}(\Delta+\Omega) - \frac{i\beta}{2}(\Delta-\Omega) = -\frac{i}{2}\left[(\alpha+\beta)\Delta + \left[(\alpha+\beta)\Omega\right]\right] = -i\Delta \Rightarrow \alpha = \beta = \frac{\Delta}{\Omega}$$

Thus
$$c_2(t) = \left(\cos\frac{\Omega t}{2} - i\frac{\Delta}{\Omega}\sin\frac{\Omega t}{2}\right)e^{-i\frac{\Delta t}{2}}$$

We can find $c_1(t) = 0$ by plugging the solution $c_2(t) = 1$ into the equation for $\dot{c}_1(t)$. Taking the result and integrating, we get (using $c_1(0) = 0$).

$$c_1(t) \;=\; i\frac{\chi^*}{2}\int_0^t c_2(t')dt' + c_1(t) \;=\; i\frac{\chi^*}{2}\int_0^t\left(\alpha e^{-\frac{i}{2}(\Delta+\Omega)t'} + \beta e^{-\frac{i}{2}(\Delta+\Omega)t'}\right)dt'$$

$$= \;-\chi^*\left[\frac{\alpha}{\Delta+\Omega}e^{-\frac{i}{2}(\Delta+\Omega)t} + \frac{\beta}{\Delta-\Omega}e^{-\frac{i}{2}(\Delta-\Omega)t} - \frac{\alpha}{\Delta+\Omega} - \frac{\beta}{\Delta-\Omega}\right]$$

It is easy to show that the time independent parts $\frac{\alpha}{\Delta+\Omega}+\frac{\beta}{\Delta-\Omega} \;=\; 0$, which leaves us with

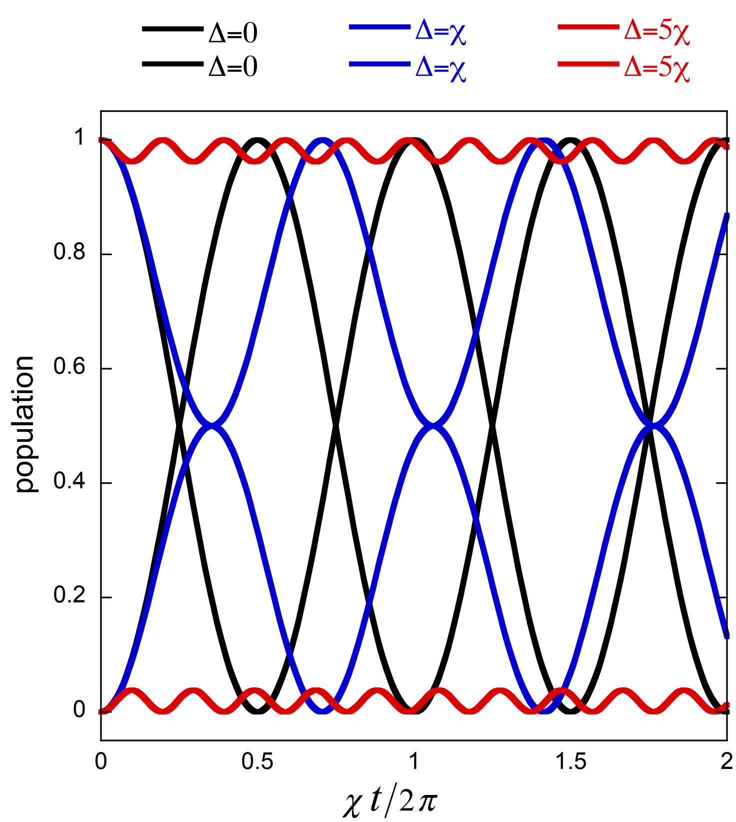
$$c_1(t) \;=\; \frac{\chi^*}{\Delta^2-\Omega^2}\left[\alpha(\Delta-\Omega)e^{-i\frac{\Omega t}{2}} + \beta(\Delta-\Omega)e^{i\frac{\Omega t}{2}}\right]e^{-i\frac{\Delta t}{2}} \;=\; i\frac{\chi^*}{\Omega}\sin\frac{\Omega t}{2}e^{-i\frac{\Delta t}{2}},$$

where the last step follows from a bit of straightforward algebra. This gives us the final, desired result:

$$\boxed{c_1(t) \;=\; i\frac{\chi^*}{\Omega}\sin\frac{\Omega t}{2}e^{-i\frac{\Delta t}{2}}, \qquad c_2(t) \;=\; \left(\cos\frac{\Omega t}{2} - i\frac{\Delta}{\Omega}\sin\frac{\Omega t}{2}\right)e^{-i\frac{\Delta t}{2}}}$$

(b) From the result in (a), we have $\left|c_1(t)\right|^2 = \frac{|\chi|^2}{\Omega^2}\sin^2\frac{\Omega t}{2}$, $\quad \left|c_2(t)\right|^2 \;=\; \cos^2\frac{\Omega t}{2} + \frac{\Delta^2}{\Omega^2}\sin^2\frac{\Omega t}{2}$

Plotting as requested, we get



II

(a) The equations of motion for the vector $\underline{c}$ corresponds to a Hamiltonian of the form

$$H_{RWA} = \hbar\begin{pmatrix} 0 & -\chi^*/2 \\ -\chi/2 & \Delta \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

Following the recipe in Cohen-Tannoudji, the eigenvalues are

$$E_{\pm} = \frac{1}{2}(H_{11}+H_{22}) \pm \frac{1}{2}\sqrt{(H_{11}+H_{22})^2+4\left|H_{12}\right|^2} = \frac{\hbar}{2}(\Delta \pm \Omega)$$

We were asked specifically to find the dressed states, i. e., the eigenstates of H_{RWA}. Again, we follow the recipe in Cohen-Tannoudji.

First, we define the <u>phase angle</u>: $\varphi = \arg(\chi)$ $(=0$, for $\chi > 0$ and real-valued)

Second, we define the <u>mixing angle</u> θ.

(i) Case $\Delta \le 0$:

We have $\tan(\theta) = -\frac{|\chi|}{\Delta} = \frac{|\chi|}{|\Delta|}$, $\theta \in [0, \frac{\pi}{2}[$, and thus $\theta = \arctan\left(\frac{|\chi|}{|\Delta|}\right)$. In that case

$$\boxed{\left|\psi_+\right\rangle = \cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}}\left|1\right\rangle + \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}}\left|2\right\rangle, \qquad \left|\psi_-\right\rangle = -\sin\frac{\theta}{2}e^{-i\frac{\varphi}{2}}\left|1\right\rangle + \cos\frac{\theta}{2}e^{i\frac{\varphi}{2}}\left|2\right\rangle}$$

(ii) Case $\Delta > 0$:

Note: This gets a bit confusing because Cohen-Tannoudji insists that we pick the mixing angle so it lies in the interval $[0, \pi[$. It helps to recall that the tangent function is periodic, $\tan(\theta) = \tan(\theta + \pi)$.

We have $\tan(\theta) = \frac{-|\chi|}{\Delta} = -\frac{|\chi|}{|\Delta|}$, $\theta \in [-\frac{\pi}{2}, 0[$, and thus the mixing angle is $\theta + \pi$, with θ defined as in part (i) above. We can use the trig identities $\cos\frac{\alpha+\pi}{2} = -\sin\frac{\alpha}{2}$, $\sin\frac{\alpha+\pi}{2} = \cos\frac{\alpha}{2}$, to neaten things up a bit. In that case

$$\boxed{\left|\psi_+\right\rangle = -\sin\frac{\theta}{2}e^{-i\frac{\varphi}{2}}\left|1\right\rangle + \cos\frac{\theta}{2}e^{i\frac{\varphi}{2}}\left|2\right\rangle, \quad \left|\psi_-\right\rangle = -\cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}}\left|1\right\rangle - \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}}\left|2\right\rangle}$$

As a simple sanity check we can note that $\langle\psi_+|\psi_+\rangle = 1$ and $\langle\psi_+|\psi_-\rangle = 0$ in both detuning regimes.

For the final part, we simplify things by choosing

Case (a): $\Delta < 0$ and $|\chi| << |\Delta|$

We use $\theta = \arctan\frac{-|\chi|}{\Delta} \approx \frac{-|\chi|}{\Delta}$. Then $\cos\frac{\theta}{2} \approx 1$ and $-\sin\frac{\theta}{2} \approx -\frac{|\chi|}{2\Delta}$, and the dressed states are

$$\boxed{|\psi_+\rangle = -\frac{|\chi|}{2\Delta}|1\rangle + |2\rangle, \qquad |\psi_-\rangle = |1\rangle - \frac{|\chi|}{2\Delta}|2\rangle}$$

That means the low-energy dressed state has mostly atomic ground state character, while the high-energy dressed state has mostly atomic excited state character – as we would expect in a regime of weak driving. The generalization to negative detuning is straightforward.

Case (b): $\Delta < 0$ and $|\chi| >> |\Delta|$

We use $\theta = \arctan(-\infty) = -\frac{\pi}{2}$. In that case the dressed states are

$$\boxed{|\psi_+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle), \qquad |\psi_-\rangle = \frac{1}{\sqrt{2}}(-|1\rangle + |2\rangle)}$$

In this regime of very strong driving the dressed state has even support on the ground and excited atomic states. Again, the generalization to negative detuning is straightforward.

Problem III

(a) We have $V_{jk_j,lk_l} = \langle j,k_j| - \hat{\vec{p}} \cdot \vec{\varepsilon} E(t) e^{ik\hat{Z}} |l,k_l\rangle = \langle j| - \hat{\vec{p}} \cdot \vec{\varepsilon} E(t)|l\rangle \langle k_j| e^{ik\hat{Z}} |k_l\rangle$

Substituting $\langle j|\hat{\vec{p}} \cdot \vec{\varepsilon} E(t)|l\rangle = \hbar\chi_{jl}(t)$ and writing the second matrix element as an overlap integral in the coordinate representation ($\psi_{P=\hbar k_q}(z) = \frac{1}{\sqrt{2\pi}} e^{ik_q z}$ and $e^{ik\hat{Z}} \to e^{ikz}$)

$$\begin{aligned} V_{jk_j,lk_l} &= \hbar\chi_{jl}(t) \int_{-\infty}^{\infty} (\tfrac{1}{\sqrt{2\pi}} e^{ik_j z})^* e^{ikz} \tfrac{1}{\sqrt{2\pi}} e^{ik_l z}\, dz = \hbar\chi_{jl}(t) \tfrac{1}{2\pi} \int_{-\infty}^{\infty} (e^{i(k_j - k_l)z})^* e^{ikz}\, dz \\ &= \hbar\chi_{jl}(t)\delta(\Delta k - k) \end{aligned}$$

where $\Delta k = k_j - k_l$.

Note: Momentum eigenstates are not physical in the sense that the corresponding plane waves are not normalizable. The same holds for the optical field. To get rid of the singularity represented by the delta function we could consider normalizable wavepackets for both the atom and the light that come arbitrarily close to plane waves. This is not important for the point we are making here.

(b) For the transition to be allowed, the momentum change of the atom when it goes from an initial state $|l\rangle$ to a final state $|j\rangle$ must be $\hbar\Delta k = \hbar k_j - \hbar k_l = \hbar k$. So in going from the ground to excited state the atom momentum must increase by $\hbar k$. If we regard the process as absorbing a photon then momentum conservation implies that said photon must carry a momentum $\hbar k$.

(c) The momentum operator is diagonal in the $|l,k_l\rangle$ basis, so $\langle \hat{P}(t)\rangle = \mathcal{P}_1\, \hbar k_1 + \mathcal{P}_2\, \hbar k_2$. Exactly at resonance the probabilities of finding the atom in the states $|1,k_1\rangle$ and $|2,k_2\rangle$ vary as

$$\mathcal{P}_1 = \cos^2(\chi t/2),\ \mathcal{P}_2 = \sin^2(\chi t/2)$$

The expectation value of the momentum is thus

$$\cos^2(\chi t/2)\hbar k_1 + \sin^2(\chi t/2)\hbar(k_1 + k) = \hbar k_1 + \sin^2(\chi t/2)\hbar k$$

Plot: