OPTI 544, Problem Set 2
Posted February 6, Due February 12

Electronic Submission only, by email to Jon Pajaud (jpajaud@email.arizona.edu)

I

In the rotating wave approximation the equation of motion for the state of a two-level atom can be reduced to the form

$$i\dot{c}_1 = -\tfrac{1}{2}\chi^* c_2 ,$$

$$i\dot{c}_2 = \Delta c_2 - \tfrac{1}{2}\chi c_1 ,$$

where $\chi = \langle 2|-\hat{\vec{p}}\cdot\vec{E}/\hbar|1\rangle$ is the (complex) Rabi frequency, and $\Delta = \omega_{21} - \omega$ is the detuning from resonance.

(a) Find the general solution to these equations. Then choose its free parameters to fit the initial conditions $c_1(t=0)=0$, $c_2(t=0)=1$.

Note: If you don't remember how to find the general solution to this kind of differential equation and therefore have a hard time answering part (a), you may instead get full credit for verifying that the solution from the class notes does in fact satisfy the equations above. Then try to guess how the solution will change for the initial conditions given here. And keep a copy of my Solution Set as a reference for the future.

(b) Plot the probabilities for finding the atom in each of the two states as a function of time, for $\Delta = 0$, $\Delta = \chi$ and $\Delta = 5\chi$. Take care to show everything to scale. (If possible, make the plots on a computer.)

II

(a) The equations of motion for the probability amplitudes c_i correspond to a time dependent Schrödinger equation governed by a time independent Hamiltonian H. Find the eigenstates (dressed states) of H as function of χ and Δ . Use the method from Chapter IV, complement B-IV in Cohen-Tannoudji, and restrict yourself to the case where χ is real and $\Delta < 0$. Derive simpler, approximate expressions for the dressed states in the limits $|\chi| << |\Delta|$ and $\chi| >> |\Delta|$.

Note: If you don't have a copy of Cohen-Tannoudji, let me know right away by email and I will get the relevant pages to you.

III

Bonus Problem - It is completely voluntary whether you decide to do this part of HW Set of HW Set 2. You will not be penalized on your score if you leave it out. But I thought it might interest some of you because it is closely analogous to the way we derived the Electric Dipole Selection Rules. I will post the solution in a day or two, and let you decide how much, if anything, of the problem you want to attempt on your own. If not, just read through my Solution Set when it posts, and pay attention to the similarity between the conservation of angular and linear momentum and the selection rules that follow.

In the following we consider a two-level atom interacting with a monochromatic plane wave $\vec{E}(t,z)=\vec{\varepsilon}E_0e^{-i(\omega t-kz)}$ of light. Allowing the atom to move along the z-axis only, the Hamiltonian for the atom is the sum of internal and kinetic energy, $\hat{H}_A=\hat{H}_0+\hat{P}^2/2M$, where $\hat{P}$ is the (center-of-mass) momentum along z. The eigenstates of $\hat{H}_A$ are joint eigenstates of internal energy and momentum,

$$|1\rangle|P_1=\hbar k_1\rangle=|1,k_1\rangle\,,\qquad |2\rangle|P_2=\hbar k_2\rangle=|2,k_2\rangle\,.$$

The electric-dipole interaction between atom and light is $\hat{V}=-\hat{\vec{p}}\cdot\vec{\varepsilon}E(t)e^{ik\hat{Z}}$, where the operator $\hat{\vec{p}}\cdot\vec{\varepsilon}E(t)$ acts solely on the internal degrees of freedom, $\langle j|\hat{\vec{p}}\cdot\vec{\varepsilon}E(t)|l\rangle=\hbar\chi_{jl}(t)$ where $j,l=1,2$. Similarly, the operator $e^{ik\hat{Z}}$ acts solely on the center-of-mass degree of freedom. We are looking for selection rules for transitions between (linear) momentum states, in a manner inspired by the way we found selection rules for dipole transitions between angular momentum states.

(a) Write down an expression for the matrix elements $V_{jk_j,lk_l}=\langle j,k_j|-\hat{\vec{p}}\cdot\vec{\varepsilon}E(t)e^{ik\hat{Z}}|l,k_l\rangle$ in terms of the matrix elements $\hbar\chi_{jl}(t)$ and an overlap integral that involves the coordinate representation of the operator $e^{ik\hat{Z}}$ and the momentum wavefunctions $\psi_{P=\hbar k_q}(z)$.

(b) What does the result in (a) tell us about the change in momentum when the atom goes from the ground state to the excited state? What does this tell us about the momentum of light?

(c) Assume the atom starts in a state $|1,k_1\rangle$, that the plane wave is turned on at $t=0$, and that it is resonant with an allowed transition to the state $|2,k_2\rangle$. Find an expression for the expectation value $\langle\hat{P}(t)\rangle$ of the momentum. Make a plot of this quantity as function of time, being careful to indicate on the axes both the timescale for the evolution and the range of momentum that is observed.

Hint: Plane waves are orthogonal functions: $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}(e^{ikz})^*\,e^{ik_0z}\,dz=\delta(k-k_0)$