OPTI 544 Solution Set 1, Spring 2024

Problem I

(a) This is entirely about the spatial dependence of the field. Thus, it suffices to show that

$$\nabla \cdot \vec{E}(\vec{r},t) = \nabla \cdot e^{i\vec{k}\cdot\vec{r}} = \frac{\partial}{\partial x} e^{i(k_x \cdot r_x)} + \frac{\partial}{\partial y} e^{i(k_y \cdot r_y)} + \frac{\partial}{\partial y} e^{i(k_z \cdot r_z)} = i\vec{k} \cdot e^{i\vec{k}\cdot\vec{r}}$$

and thus $\vec{E}(\vec{r},t) = \hat{\varepsilon} E_0 e^{-i(\omega t - \vec{k}\cdot\vec{r})}$ is transverse if and only if $\vec{k} \cdot \hat{\varepsilon} = 0$

(b) Equation of motion: $$\frac{d^2}{dt^2}\mathbf{x} + 2\beta \frac{d}{dt}\mathbf{x} + \omega_0^2 \mathbf{x} = \frac{e}{m} \vec{\varepsilon} E_0 e^{-i(\omega t - kz)}$$

Plug in trial solution $\mathbf{x}(t) = \vec{a} e^{-i(\omega t - kz)}$ where $\vec{a}$ is constant and to be determined:

$$\left[\frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega_0^2\right] \vec{a} e^{-i(\omega t - kz)} = \left[-\omega^2 - 2i\beta\omega + \omega_0^2\right] \vec{a} e^{-i(\omega t - kz)} = \frac{e}{m} \vec{\varepsilon} E_0 e^{-i(\omega t - kz)}$$

Cancelling out the exponential and rearranging terms gives us

$$\left[\omega_0^2 - \omega^2 - 2i\beta\omega\right] \vec{a} = \frac{e}{m} \vec{\varepsilon} E_0 \;\Rightarrow\; \vec{a} = \vec{\varepsilon} \frac{(e/m) E_0}{\omega_0^2 - \omega^2 - 2i\beta\omega}$$

(c) We have $$\mathbf{p} = \alpha \mathbf{E} = \alpha \vec{\varepsilon} E_0 e^{-i(\omega t - kz)} = |\alpha| e^{i\phi} \vec{\varepsilon} E_0 e^{-i(\omega t - kz)}$$

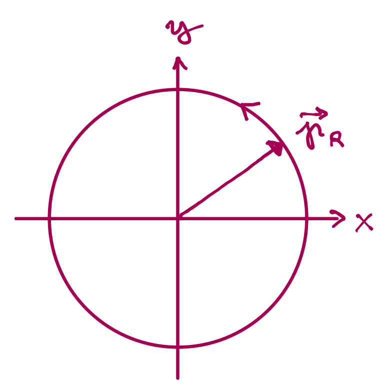
Taking E_0 as real, the complex polarizability leads to a detuning-dependent phase lag between $\mathbf{p}$ and $\vec{\varepsilon}$. To make the math a little less cumbersome, we can shift the origin of the time axis by an amount δt, so that $e^{i\phi} e^{-i\omega\delta t} = 1$. This allows us to once again write $\mathbf{p} = |\alpha| \; \vec{\varepsilon} \; E_0 e^{-i(\omega t - kz)}$, where $\mathbf{p} \parallel \vec{\varepsilon}$, and the motion of $\mathbf{p}$ is identical to the motion of $\vec{\varepsilon}$ (except for the phase lag).

Let $\vec{\varepsilon} = \vec{\varepsilon}_x$ and let $\mathbf{p}_R$ be the physical dipole. Then

$$\mathbf{p}_R = \mathrm{Re}[\mathbf{p}] \propto \mathrm{Re}[\vec{\varepsilon}_x e^{-i(\omega t - kz)}] = \vec{\varepsilon}_x \cos(\omega t - kz)$$

This is a dipole oscillating along $\vec{\varepsilon}_x$ with frequency ω.
Next, assume

$$\vec{\varepsilon} = \vec{\varepsilon}_+ \;\Rightarrow\; \mathbf{p}_R = -\tfrac{1}{\sqrt{2}} \mathrm{Re}[\vec{\varepsilon}_x e^{-i(\omega t - kz)} + i\vec{\varepsilon}_y e^{-i(\omega t - kz)}]$$
$$\propto \vec{\varepsilon}_x \cos(\omega t - kz) + \vec{\varepsilon}_y \sin(\omega t - kz)$$



From this we see that $\mathbf{p}_R$ rotates counter-clockwise in the x-y plane when viewed from the $+z$ direction, with angular frequency ω. The rotation of $\mathbf{p}_R$ lags the rotation of $\mathbf{E}$ by the phase ϕ.

Problem II

(a) Let $\begin{cases} n_R = 1 + \dfrac{Ne^2}{4\varepsilon_0 m\omega}\dfrac{\Delta}{\Delta^2+\beta^2} \\ n_I = \dfrac{Ne^2}{4\varepsilon_0 m\omega}\dfrac{\beta}{\Delta^2+\beta^2} \end{cases}$, where $\omega_0 - \omega$.

The phase delay induced by the optical medium at $\Delta = 0$ is proportional to the real index of refraction, here $n_R = 1$. There is thus no extra phase delay relative to vacuum.

Extinction coefficient:

$$a = \frac{2\omega}{c} n_I = \frac{Ne^2}{2\varepsilon_0 m\beta c}$$

$$= \frac{10^{16} m^{-3} (1.602\times10^{-19}\, C)^2}{2\times 8.85\times10^{-12}\,\frac{F}{m}\times 9.11\times10^{-31} kg \times 1.5\times10^{7} s^{-1}\times 3\times10^{8} ms^{-1}} = 3536.9 m^{-1}$$

Transmission: $T = e^{-al} = e^{-0.05m\times 3536.9m^{-1}} = 1.57\times10^{-77}$. The cell is **totally opague!**

(b) The gas contains subsets of atoms (velocity classes). Consider an atom moving with velocity v along the axis of wave propagation, such that the apparent resonance frequency is $\omega = \omega_0 + kv$ in the lab frame.

The probability distribution over velocity is $P(v) = \frac{1}{\sqrt{2\pi\sigma_v^2}} e^{-v^2/2\sigma_v^2}$, where

$$\sigma_v = \sqrt{\frac{k_B T}{M}} = \sqrt{\frac{1.38\times10^{-23}\,\frac{J}{K}\times 300K}{2.21\times10^{-25} kg}} = 136\ ms^{-1}$$

And the corresponding probability distribution over frequencies is

$$P(\omega) = P\left(v = \frac{\omega-\omega_0}{k}\right) = \frac{1}{\sqrt{2\pi\sigma_\omega^2}} e^{-\omega^2/2\sigma_\omega^2} ,$$

where $$\sigma_\omega = \frac{2\pi}{\lambda}\sigma_v = \frac{2\pi}{894\times10^{-9} m}\times 136\ ms^{-1} = 9.622\times10^{8} s^{-1} = 2\pi\times 153.1\ \text{MHz}$$

Now let the plane wave frequency be ω_L. The number density of atoms with apparent resonance frequency ω in the lab frame is $N(\omega) = N\times P(\omega)$, and each velocity class contributes to the total complex index of refraction according to number density and detuning. Thus, we have

$$a(\omega_L) = \int_{-\infty}^{\infty} P(\omega)\, a(\omega - \omega_L)\, d\omega$$

where $$a(\omega-\omega_L)=\frac{Ne^2}{2\varepsilon_0 mc}\frac{\beta}{(\omega-\omega_L)^2+\beta^2}.$$

Note: $a(\omega_L)$ is maximum when ω_L lies at the peak of the frequency distribution $P(\omega)$, i. e., when $\omega_L=\omega_0$. Thus, to find the minimum transmission we need to compute

$$a(\omega_0)=\int_{-\infty}^{\infty} P(\omega)\, a(\omega-\omega_0)\, d\omega$$

Noting that $\beta<<\sigma_\omega$, we can approximate $a(\omega-\omega_0)$ with a δ-function in the integral, which gives us

$$a(\omega_0)=\frac{1}{\sqrt{2\pi\sigma_\omega^2}}\frac{Ne^2}{2\varepsilon_0 mc}\int_{-\infty}^{\infty} e^{-(\omega_0-\omega)^2/2\sigma_\omega^2}\frac{\beta}{(\omega-\omega_0)^2+\beta^2}\, d\omega$$

$$=\frac{1}{\sqrt{2\pi\sigma_\omega^2}}\frac{Ne^2}{2\varepsilon_0 mc}\int_{-\infty}^{\infty} e^{-(\omega_0-\omega)^2/2\sigma_\omega^2}\,\pi\,\delta(\omega-\omega_0)\, d\omega$$

$$=\frac{1}{\sqrt{2\pi\sigma_\omega^2}}\frac{Ne^2\pi}{2\varepsilon_0 mc}=69.10m^{-1}$$

Minimum Transmission: $T=e^{-a(\omega_0)l}=0.031\ \sim\ 3\%$

This is still a small fraction of the light, but the cell is not completely opaque. Besides, small variations in the total number density of atoms and the temperature can make a significant difference.

Problem III

We model aluminum as a free electron gas, which is approximated by a collection of electron oscillators with $\omega_0 \to 0$.

In that case the medium is transparent above the *plasma frequency* $\omega_P = \sqrt{\dfrac{Ne^2}{\varepsilon_0 m}}$

First we estimate *N*. The number density of Aluminum atoms is

$$N_{Al} = \frac{2700\,kg\,m^{-3}}{4.48\times10^{-26}kg} = 6.03\times10^{28}\,m^{-3} \;\Rightarrow\; N = 3N_{Al} = 1.81\times10^{29}\,m^{-3}$$

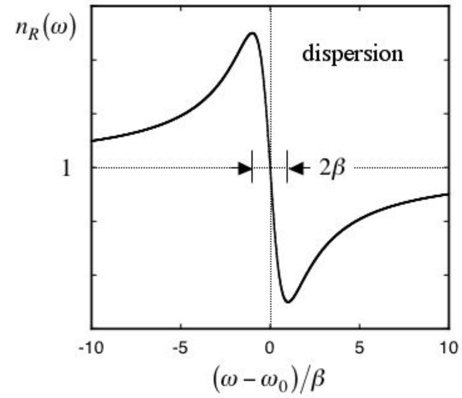
Thus $$\omega_P = 2.40\times10^{16}\,s^{-1} \;\Rightarrow\; \lambda_P = \frac{2\pi c}{\omega_P} = 78.53nm.$$

Our model suggests aluminum is reflective for wavelengths above λ_P.

In practice aluminum is a good reflector above $200nm$. The exact behavior of the reflectivity depends on the oxidation of the metal surface, among other things. And of course aluminum is not transparent below λ_P, due to its non-zero conductivity at optical frequencies. "Transparency" is an artifact of our electron oscillator model because we ignored losses when setting $\beta \sim 0$.

Problem IV

(a) From the notes on the electron oscillator model:



Thus $n_R > 1$ occurs when $\omega < \omega_0$.

(b) From the same notes, we have in general

$$n(\omega)^2 = 1 + \frac{Ne^2}{m\varepsilon_0} \frac{(\omega_0^2 - \omega^2) + 2i\beta\omega}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

The index of refraction is real-valued when $|\beta\omega| << |\omega_0^2 - \omega^2| = |(\omega_0 + \omega)(\omega_0 - \omega)|$, i. e., in the large detuning limit. In that case

$$n(\omega)^2 = n_R(\omega)^2 = 1 + \frac{Ne^2}{m\varepsilon_0} \frac{1}{\omega_0^2 - \omega^2}$$

(c) The derivative is

$$\frac{dn_R}{d\omega} = \frac{d}{d\omega}\left(1 + \frac{Ne^2}{m\varepsilon_0}\frac{1}{\omega_0^2 - \omega^2}\right)^{1/2} = \frac{1}{2}\left(1 + \frac{Ne^2}{m\varepsilon_0}\frac{1}{\omega_0^2 - \omega^2}\right)^{-1/2} \times \frac{Ne^2}{m\varepsilon_0}\frac{2\omega}{(\omega_0^2 - \omega^2)^2}$$

$$\Rightarrow\ n_R(\omega)\frac{dn_R}{d\omega} = \frac{Ne^2}{m\varepsilon_0}\frac{\omega}{(\omega_0^2 - \omega^2)^2}$$

Combining results from (b) and (c) we get

$$\kappa \ = \ \frac{n_R(\omega)}{n_R(\omega)^2 - 1}\frac{dn_R}{d\omega} \ = \ \frac{\omega / (\omega_0^2 - \omega^2)^2}{1/(\omega_0^2 - \omega^2)^2} \ = \ \frac{\omega}{\omega_0^2 - \omega^2}$$

$$\Rightarrow\ \kappa\omega_0^2 - \kappa\omega^2 = \omega\ \Rightarrow\ \omega_0 = \sqrt{\frac{\omega(1+\kappa\omega)}{\kappa}}$$

Now $\qquad \kappa \ = \ \dfrac{1.458}{1.458^2 - 1} \times 6.36 \times 10^{-18} s \ = \ 8.237^{-18} s$

Then $\qquad \omega_0 = \sqrt{\dfrac{3.14 \times 10^{15} s^{-1}(1 + 8.237 \times 10^{-18} s\ \times 3.14 \times 10^{15} s^{-1})}{8.237 \times 10^{-18} s}} \ = \ 1.978 \times 10^{16} s^{-1}$

and $\qquad \lambda_0 = \dfrac{2\pi c}{\omega_0} = 95.3 nm$.