OPTI 544, Problem Set 1
Posted January 24, Due February 2

Electronic Submission only, by email to Jon Pajaud (jpajaud@email.arizona.edu)

I

(a) Show that a plane wave $\vec{E}(\vec{r},t) = \hat{\varepsilon} E_0 e^{-i(\omega t - \vec{k}\cdot r)}$ is transverse, $\nabla \cdot \vec{E}(\vec{r},t) = 0$, if $\hat{\varepsilon} \perp \vec{k}$.

(b) Given a damped electron oscillator and a driving field $\vec{E}(z,t) = \hat{\varepsilon} E_0 e^{-i(\omega t - kz)}$, write down the equation of motion for the electron displacement $\vec{x}(t)$. Look for a solution of the form $\vec{x}(t) = \vec{a} e^{-i(\omega t - kz)}$ and derive an expression for $\vec{a}$.

(c) Describe the nature of the motion of the atomic dipole for $\hat{\varepsilon} = \hat{x}$ (linear polarization), and for $\hat{\varepsilon}_+ = -(\hat{x} + i\hat{y})/\sqrt{2}$ (circular polarization).

II

A cell of length $l = 5cm$ contains Cs vapor at a density $N = 10^{10} cm^{-3}$. We assume that the Cs atoms behave like classical electron oscillators with a damping rate $\beta = 1.5 \times 10^7 s^{-1}$, and a transition wavelength of $\lambda = 894 nm$ for an atom at rest.

(a) Assuming that atoms in the vapor are at rest, calculate the excess phase delay relative to propagation in vacuum, and the transmission $T = I_{out}/I_{in}$ for an on-resonance plane wave, $\omega = \omega_0$, passing through the cell.

We now take into account thermal motion in the gas. This leads to *Doppler broadening*. (If you are not familiar with the concept, see the note at the end of this HW Set.

(b) The temperature of the vapor is $T = 300$ K, and the mass of the Cs atom is $M = 2.21 \times 10^{-25}$ kg. Find the probability distribution $P(\omega)$ for atoms whose resonance frequency is ω_0 in the rest frame, and Doppler shifted to frequency ω in the lab frame due to their velocity along the plane wave direction of propagation. Noting that this Doppler broadened frequency distribution is much wider than the natural linewidth, find the minimum value of the transmission for a plane wave passing through the cell.

III

The density of aluminum metal is $2.70 g/cm^3$. Assuming that each atom contributes its three valence electrons to the "electron gas", at what wavelengths would you expect aluminum metal to be reflective.

Some numbers:	$c = 2.998 \times 10^8 \, m/s$	$e = 1.602 \times 10^{-19} C$
	$\mu_0 = 4\pi \times 10^{-7} \, N/A^2$	$m_e = 9.11 \times 10^{-31} kg$
	$\varepsilon_0 = 8.854 \times 10^{-12} \, F/m$	$m_p = 1.672 \times 10^{-27} kg$
	$\hbar = 1.055 \times 10^{-34} \, Js$	$m_{Al} = 4.48 \times 10^{-26} kg$

IV

Fused silica has a purely real index of refraction $n(\omega) = n_R(\omega) = 1.458$ at a frequency $\omega = 3.14 \times 10^{15} s^{-1}$ (wavelength $\lambda = 0.6 \mu m$). The *chromatic dispersion* at this frequency is $dn_R(\omega)/d\omega = 6.36 \times 10^{-18} s$.

(a) It is common to ascribe the simultaneous transparency and large index of refraction for visible light in fused silica to a strong, far-off resonance optical transition. Given the value of $n_R(\omega)$ above, do you expect the resonance frequency ω_0 to lie above or below ω?

(b) Write down (don't derive) a general expression for $n(\omega)^2$, as well as a simplified expression valid when this quantity is purely real, $n(\omega)^2 = n_R(\omega)^2$, as in our example here. Note: you cannot use the near-resonance or weak polarizability approximations in this case.

(c) Based on your result in (b), find a simple expression for the quantity $n_R(\omega)\dfrac{dn_R(\omega)}{d\omega}$.

(d) Use your results from (b) & (c) to find an expression for the quantity $\kappa = \dfrac{n_R(\omega)}{n_R(\omega)^2 - 1}\dfrac{dn_R(\omega)}{d\omega}$

that depends only on ω, ω_0. Then, given that the value of κ is known from the values of $n_R(\omega)$ and $dn_R(\omega)/d\omega$ listed above, calculate the resonance wavelength λ_0 in nanometer. (Your final answer must be a number!)

Note on Doppler Broadening

A gas at finite temperature can be regarded as an ensemble of atoms moving at different velocities (velocity classes). Consider an atom moving with velocity v along the axis of wave propagation, such that the apparent resonance frequency is $\omega = \omega_0 + kv$ in the lab frame.

The probability distribution over velocity is $P(v) = \dfrac{1}{\sqrt{2\pi\sigma_v^2}} e^{-v^2/2\sigma_v^2}$, where

$$\sigma_v = \sqrt{\frac{k_B T}{M}} = \sqrt{\frac{1.38 \times 10^{-23} \frac{J}{K} \times 300K}{2.21 \times 10^{-25} kg}} = 136\ ms^{-1}$$

and the corresponding probability distribution over frequencies is

$$P(\omega) = P\left(v = \frac{\omega - \omega_0}{k}\right) = \frac{1}{\sqrt{2\pi\sigma_\omega^2}} e^{-\omega^2/2\sigma_\omega^2},$$

where $$\sigma_\omega = \frac{2\pi}{\lambda}\sigma_v = \frac{2\pi}{894 \times 10^{-9} m} \times 136\ ms^{-1} = 9.622 \times 10^8 s^{-1} = 2\pi \times 153.1 \text{ MHz}$$