Light-Matter Interaction

Hierarchy of Sophistication:

- Classical — Classical light, classical matter
- Semiclassical — Classical light, quantum matter
- Quantum — Quantum light, quantum matter

Possible attitudes:

- Purist — Most complete description possible
- Minimalist — Quantum only when necessary
- Pragmatic — Quantum or classical, based on what is simplest and still works

OPTI 544: All of the above in turn

Classical Theory of Light-Matter Interaction

Self-consistent, fully classical description

$$\text{Electromagnetic Field} \rightleftarrows \text{Atom/Molecule/Solid}$$

Motivation: We will

- Develop Concepts $\alpha(\omega), n, \chi$
- Develop Intuition
- Classical is often adequate, sometimes accurate
- A Quantum Theory has classical limits ⇨ Identify/understand regime of validity
- The Classical description is a useful starting point for Nonlinear and Quantum Optics

Light-Matter Interaction

The Electromagnetic Field: Basic Eqs. in SI Units

Maxwell's eqs.

(no free charges, currents ⇨ dielectrics)

(i) $\nabla \cdot \vec{D} = \rho = 0$ $\vec{D}$: Dielectric displacement

(ii) $\nabla \cdot \vec{B} = 0$ $\vec{B}$: Magnetic induction

(iii) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\vec{E}$: Electric field

(iv) $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$ $\vec{H}$: Magnetic field

Material Response:

(v) $\vec{B} = \mu_0 \vec{H} + \vec{M}$ ← Non-magnetic ⇨ $\vec{M} = 0$

(vi) $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ ← Info about response in dipole moment density (polarization density)

We need equations that describe:

- the behavior of $\vec{E}$ for given $\vec{P}$
- the medium response $\vec{P}$ for given $\vec{E}$

Wave Equation:

Take curl of (iii), then use (iv)

$$\nabla \times (\nabla \times \vec{E}) = -\nabla \times \frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \vec{B}) = -\mu_0 \frac{\partial^2 \vec{D}}{\partial t^2}$$

Next, use the identity

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

to obtain $\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu_0 \frac{\partial^2 \vec{D}}{\partial t^2}$

Finally, let $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$, use $\varepsilon_0 \mu_0 = \frac{1}{c^2}$,

and rearrange to obtain

$$\boxed{\nabla^2 \vec{E} - \nabla(\nabla \cdot \vec{E}) - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\varepsilon_0 c^2} \frac{\partial^2 \vec{P}}{\partial t^2}}$$

This is the Wave Equation, still exact in this form

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Transverse Fields

Definition: a field for which $\nabla \cdot \vec{E} = 0$ is Transverse

Example: a plane wave, $\vec{E}(\vec{r},t) = \vec{E}(t)\, e^{i\vec{k}\cdot\vec{r}}$,

where $\vec{E}(t) \perp \vec{k}$, is transverse.

The physical field is $Re[\vec{E}(\vec{r},t)]$

For transverse fields the wave equation simplifies to

$$\boxed{\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\vec{P}}$$

This version of the wave equation can be a poor approximation in non-isotropic media!

Isotropic Media

Absent a preferred direction, the induced $\vec{P}$ must be parallel to the driving field $\vec{E}$

Regime of Linear response, steady state:

$$\vec{D}(t) = \varepsilon_0 \vec{E}(t) + \vec{P}(t) \quad \text{where} \quad \vec{P}(t) = R\cdot\vec{E}(t)$$

Constant, same units as ε_0, but $R \gg \varepsilon_0$

Regime of Linear response, transient case:

$$\vec{D}(t) = \varepsilon_0 \vec{E}(t) + \int_{-\infty}^{t} dt' R(t-t')\vec{E}(t')$$

where the *response function* $R(t-t')$ is a scalar and we have $R(\tau) = 0$ for $\tau < 0$

Take divergence on both sides and use M.E. (1)

$$\nabla\cdot\vec{D}(t) = \varepsilon_0 \nabla\cdot\vec{E}(t) + \int_{-\infty}^{t} dt' R(t-t')\nabla\cdot\vec{E}(t') = 0$$

and $\nabla\cdot\vec{E}(t) = -\int_{-\infty}^{t} dt' R(t-t')\nabla\cdot\vec{E}(t')$ for all t

It follows that $\nabla\cdot\vec{E}(t) = 0$ for all t,

OR $\quad R(\tau) = -2\delta(\tau)$

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OR $R(\tau) = -2\delta(\tau)$

Note: if $R(\tau) \propto \delta(\tau)$ **(instantaneous response) then**

$$\varepsilon_0 \int_{-\infty}^{t} dt' R(t-t') \vec{E}(t') = \varepsilon_0 \chi \vec{E}(t)$$

susceptibility

The case $R(\tau) = -2\delta(\tau)$ **is an example of negative susceptibility,** $\chi < 0$ **, which only occurs in certain engineered metamaterials.**

Electric fields are transverse in linear, isotropic dielectric media

(including the vacuum)

Wave Equation in free space

$$\boxed{\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} = 0}$$

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Monochromatic trial solution $\vec{E}(\vec{r},t) = \vec{E}_0(\vec{r}) e^{-i\omega t}$

$$\nabla^2 \vec{E}_0(\vec{r}) e^{-i\omega t} + \frac{\omega^2}{c^2} \vec{E}_0(\vec{r}) e^{-i\omega t} = 0$$

Equation for the spatial component alone:

$$\nabla^2 \vec{E}_0(\vec{r}) + |\vec{k}|^2 \vec{E}_0(\vec{r}) = 0 \, , \ |\vec{k}| = \omega/c$$

Plane wave solutions

$$\boxed{\vec{E}_0(\vec{r}) = \sum E_0 e^{i\vec{k}\cdot\vec{r}} \, , \ |\vec{k}| = \omega/c}$$

<u>Optical Cavities</u>: Here we need to solve the wave equation subject to boundary conditions. See, e. g., Millony & Eberly for examples such as rectangular cavities, Fabryt-Perot etalons, and spherical mirror resonators.

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Wave Equation in Fourier Space:

In Configuration Space:

$$\nabla^2 \vec{E}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\vec{P}(\vec{r},t)$$

In Fourier Space:

$$\boxed{k^2 \vec{E}(\vec{k},\omega) - \frac{\omega^2}{c^2}\vec{E}(\vec{k},\omega) = \frac{\omega^2}{\varepsilon_0 c^2}\vec{P}(\vec{k},\omega)}$$

<u>Note</u>: In the Fourier domain the wave equation is purely algebraic – there are no derivatives or integrals. This becomes important later in the course when we quantize the electromagnetic field.

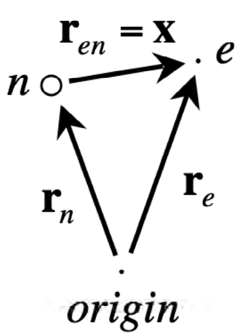
Light-Matter Interaction

Theory of Atomic Response

So far, we have a model for the field. Next, we need a model of how the constituents of the medium responds to the field.

This will allow us to find the polarization density $\vec{P}$ as function of the field $\vec{E}$

Classical "atom"



Simple model:
nucleus + electron

Lorentz Force

$$\boxed{\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)}$$

~ 0 if non-relativistic

Newton:

$$\boxed{\begin{aligned} &\text{(i)} \quad m_n \frac{d^2}{dt^2}\vec{r}_n(t) = -e\,\vec{E}(\vec{r}_n, t) - \vec{F}_{en}(\vec{r}_{en}, t) \\ &\text{(ii)} \quad m_e \frac{d^2}{dt^2}\vec{r}_e(t) = e\,\vec{E}(\vec{r}_e, t) + \vec{F}_{en}(\vec{r}_{en}, t) \end{aligned}}$$

This is a standard 2-body problem which we can re-cast as in terms of relative and COM motion.

We define:

$$\vec{x} = \vec{r}_{en} = \vec{r}_e - \vec{r}_n \qquad m = \frac{m_e m_n}{m_e + m_n} \sim m_e$$

$$\vec{R} = \frac{m_e \vec{r}_e + m_n \vec{r}_n}{M} \qquad M = m_e + m_n \sim m_n$$

$\vec{x}$ Relative coord. $\quad$ m Reduced mass

$\vec{R}$ Center-of-mass $\quad$ M Total mass

Light-Matter Interaction

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Sub into (i), (ii) and rewrite:

$$M\frac{d^2}{dt^2}\vec{R} = e\left[\vec{E}\left(\vec{R}+\frac{m_n}{M}\vec{x},t\right) - \vec{E}\left(\vec{R}-\frac{m_e}{M}\vec{x},t\right)\right]$$

$$m\frac{d^2}{dt^2}\vec{x} = \frac{e}{2}\left[\vec{E}\left(\vec{R}+\frac{m_n}{M}\vec{x},t\right) + \vec{E}\left(\vec{R}-\frac{m_e}{M}\vec{x},t\right)\right] + \vec{F}_{en}(\vec{x}) + \frac{1}{2}(m_n - m_e)\frac{d^2}{dt^2}\vec{R}$$

Basic result, no approximations !

Milloni & Eberly, main text ⇨ **Set** $\vec{R} \approx \vec{r}_n$, $\vec{x} \approx \vec{r}_{en}$

Throw away eq. for $\vec{R}$

<u>Electric Dipole approximation</u>

Atomic dimensions		Optical Wavelength
$\lvert\vec{x}\rvert \sim 1\text{Å}$	$\ll$	$\lambda \sim 10^4\text{Å}$

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<u>EDA</u>: the field is <u>nearly constant</u> on the scale of an atom

Good approximation: 1st order expansion in $\vec{x}$

$$\vec{E}\left(\vec{R}-\frac{m_e}{M}\vec{x},t\right)\approx\vec{E}(\vec{R},t)-\frac{m_e}{M}(\vec{x}\cdot\nabla)\vec{E}(\vec{R}(t))$$

$$\vec{E}\left(\vec{R}+\frac{m_e}{M}\vec{x},t\right)\approx\vec{E}(\vec{R},t)+\frac{m_e}{M}(\vec{x}\cdot\nabla)\vec{E}(\vec{R}(t))$$

$$M\frac{d^2}{dt^2}\vec{R}\approx e(\vec{x}\cdot\nabla)E(\vec{R},t) \quad \text{COM}$$

$$m\frac{d^2}{dt^2}\vec{x} = e\vec{E}(\vec{R},t)+\frac{m_n-m_e}{M}e(\vec{x}\cdot\nabla)\vec{E}(\vec{R},t) + \vec{F}_{en}(\vec{x}) \quad \text{Rel. Coord.}$$

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Physical Interpretation:

$\vec{p} = e\vec{x}$: **electric dipole moment of the atom**

The Eqs. Of motion can then be recast as

$$M\frac{d^2}{dt^2}\vec{R} \approx (\vec{p}\cdot\nabla)\vec{E}(\vec{R},t) = \vec{F} = -\nabla_R V(\vec{x},\vec{R},t)$$

$$m\frac{d^2}{dt^2}\vec{x} = e\vec{E}(\vec{R},t) + \vec{F}_{en}(\vec{x}) = -\nabla_x V(\vec{x},\vec{R},t)$$

where $V(\vec{x},\vec{r},t) = -\vec{p}\cdot\vec{E}(\vec{r},t)$

electric-dipole interaction

Note: The COM Eq. is the foundation for a range of laser Atom Traps and Optical Tweezers. We will not explore this further in OPTI 544 lectures, but good review articles can be found in the published literature.

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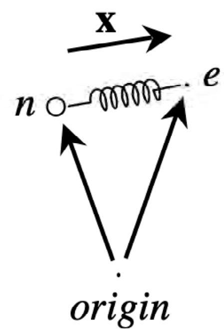
The Electron Oscillator/Lorentz Oscillator

Simple model w/a harmonically bound electron:

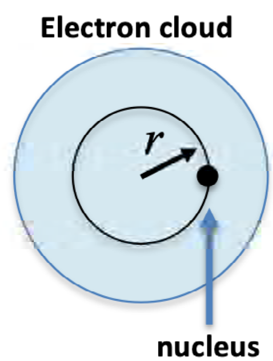


$\vec{F}_{en} = -m\omega_0^2\vec{x}$

resonance frequency

This is meant as a model of the atomic <u>response</u>, not a model of the atom itself.

<u>Nevertheless</u>: QM suggest the atom consists of a point-like nucleus and a spherical electron cloud



Force from charge inside r as if entire charge was at the center

Force from charge outside r is zero

Force $F \propto \frac{r^3}{r^2} \propto r$

harmonic restoring force

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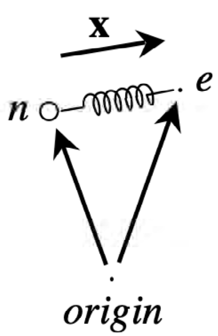

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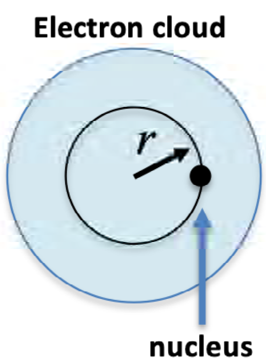

Force from charge inside r as if entire charge was at the center

Force from charge outside r is zero

Force $F \propto \dfrac{r^3}{r^2} \propto r$

harmonic restoring force

Now substitute $\vec{F}_{en} = -m\omega_0^2 \vec{x}$ into eq. for $\vec{x}$

$$\boxed{\frac{d^2}{dt^2}\vec{x} + \omega_0^2 \vec{x} = \frac{e}{m}\vec{E}(\vec{R},t)}$$

Combine with $\vec{P} = N\vec{p}$, $\vec{p} = e\vec{x}$ where N is the number density of atoms. This relates the macroscopic $\vec{P}$ to the microscopic $\vec{x}$

We now have

Maxwell's Equations
The Lorentz model

Maxwell-Lorentz Equations
We can seek self-consistent solutions to wave propagation

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