

Wigner-Weisskopf Theory of Spontaneous Decay

Today: Decay of atomic excited state
due to interaction with the
quantum electromagnetic field

Expand

$$|\Psi_I(t)\rangle = C_{2,0}(t) |2,0\rangle + \sum_{\vec{k}, \lambda} C_{1,1}_{\vec{k}, \lambda}(t) |1,1_{\vec{k}, \lambda}\rangle$$

Setup

Hamiltonian: (Schrödinger Picture)

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} = \sum_{\vec{k}} \hbar \omega_{\vec{k}} \hat{a}_{\vec{k}}^+ \hat{a}_{\vec{k}} + \frac{1}{2} \hbar \omega_2 \hat{\sigma}_z + \sum_{\vec{k}} \hbar (g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}}^* \hat{\sigma}_- \hat{a}_{\vec{k}}^+)$$

Hamiltonian: (Interaction Pict., Res. Approx.)

$$\hat{H}_I(t) = \sum_{\vec{k}, \lambda} \hbar g_{\vec{k}, \lambda} \hat{\sigma}_+ \hat{a}_{\vec{k}, \lambda} e^{i(\omega_u - \omega_{\vec{k}})t} + H.C.$$

$$S.E.: i\hbar \frac{d}{dt} |\Psi_I(t)\rangle = \hat{H}_I(t) |\Psi_I(t)\rangle$$

$$\dot{C}_{2,0}(t) = -i \sum_{\vec{k}, \lambda} g_{\vec{k}, \lambda} e^{i(\omega_u - \omega_{\vec{k}})t} C_{1,1}_{\vec{k}, \lambda}(t)$$
$$\dot{C}_{1,1}_{\vec{k}, \lambda}(t) = -i g_{\vec{k}, \lambda}^* e^{-i(\omega_u - \omega_{\vec{k}})t} C_{2,0}(t)$$

infinite # of these

Formal Solution:

$$C_{1,1}_{\vec{k}, \lambda}(t) = -i g_{\vec{k}, \lambda}^* \int_0^t e^{-i(\omega_u - \omega_{\vec{k}})t'} C_{2,0}(t') dt'$$

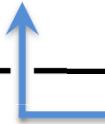
Wigner-Weisskopf Theory of Spontaneous Decay

Expand

$$|\Psi(t)\rangle = C_{2,0}(t) |2,0\rangle + \sum_{\vec{k}, \lambda} C_{1,1_{\vec{k}, \lambda}}(t) |1,1_{\vec{k}, \lambda}\rangle$$



$$\dot{C}_{2,0}(t) = -i \sum_{\vec{k}, \lambda} g_{0_{\vec{k}, \lambda}} e^{i(\omega_u - \omega_{\vec{k}})t} C_{1,1_{\vec{k}, \lambda}}(t)$$
$$\dot{C}_{1,1_{\vec{k}, \lambda}}(t) = -i g_{\vec{k}, \lambda}^* e^{-i(\omega_u - \omega_{\vec{k}})t} C_{2,0}(t)$$



infinite # of these

$$\dot{C}_{2,0}(t) = - \sum_{\vec{k}, \lambda} |g_{0_{\vec{k}, \lambda}}|^2 \int_0^t e^{i(\omega_u - \omega_{\vec{k}})(t-t')} C_{2,0}(t') dt'$$

Time Dep. Perturbation Theory:

Study short-time limit, $C_{2,0}(t) \sim 1$

→ Fermi's Golden Rule

No good for this problem!

Formal Solution:

$$C_{1,1_{\vec{k}, \lambda}}(t) = -i g_{\vec{k}, \lambda}^* \int_0^t e^{-i(\omega_u - \omega_{\vec{k}})t'} C_{2,0}(t') dt' \quad \rightarrow$$

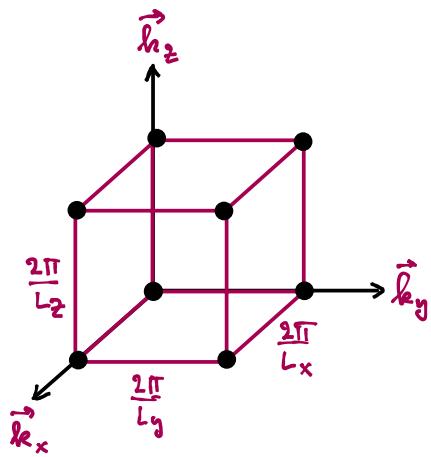
Wigner-Weisskopf Theory of Spontaneous Decay

$$\dot{C}_{2,0}(t) = - \sum_{\vec{k}, \lambda} |g_{\vec{k}, \lambda}|^2 \int_0^t e^{i(\omega_{\vec{k}} - \omega_{\vec{k}})(t-t')} C_{2,0}(t') dt'$$

Quantization in a box w/periodic B. C.

$$\rightarrow \vec{k}_i = n_i \frac{2\pi}{L}, \quad n \text{ integer}$$

In \vec{k} space the modes form a grid:



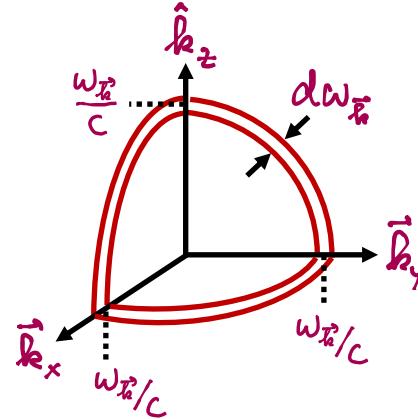
$$\begin{aligned} & \text{1 mode} \\ & \left(\frac{2\pi}{L_x} \right) \left(\frac{2\pi}{L_y} \right) \left(\frac{2\pi}{L_z} \right) \\ & = \frac{V}{(2\pi)^3} = \mathcal{D}(\vec{k}) \end{aligned}$$

Density of Modes

Convert sum to integral over modes:

$$\begin{aligned} \sum_{\vec{k}} & \rightarrow \int d^3 \vec{k} \mathcal{D}(\vec{k}) = \int \vec{k}^2 d(\hat{\vec{k}}) d\vec{k} \mathcal{D}(\vec{k}) \\ & = \int d(\hat{\vec{k}}) d\omega_{\vec{k}} \frac{\omega_{\vec{k}}^2}{C^3} \mathcal{D}(\vec{k}) = \int d(\hat{\vec{k}}) d\omega_{\vec{k}} \mathcal{D}(\omega_{\vec{k}}) \end{aligned}$$

where $\hat{\vec{k}} = \vec{k}/k$ is a unit vector along \vec{k} and



$$\mathcal{D}(\omega_{\vec{k}}) = \frac{V}{(2\pi)^3} \frac{\omega_{\vec{k}}^2}{C^3}$$

mode density in shell of
 \vec{k} - space of radius $\omega_{\vec{k}}/C$

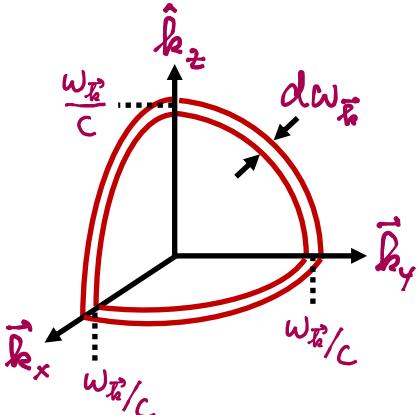
Wigner-Weisskopf Theory of Spontaneous Decay

$$\dot{c}_{2,0}(t) = - \sum_{\vec{k}, \lambda} |\tilde{g}_{\vec{k}, \lambda}|^2 \int_0^t e^{i(\omega_{\text{R}} - \omega_{\vec{k}})(t-t')} c_{2,0}(t') dt'$$

Convert sum to integral over modes:

$$\begin{aligned} \sum_{\vec{k}} &\rightarrow \int d^3 \vec{k} D(\vec{k}) = \int k^2 d(\hat{k}) dk D(\vec{k}) \\ &= \int d(\hat{k}) d\omega_{\vec{k}} \frac{\omega_{\vec{k}}^2}{C^3} D(\vec{k}) = \int d(\hat{k}) d\omega_{\vec{k}} D(\omega_{\vec{k}}) \end{aligned}$$

where $\hat{k} = \vec{k}/k$ is a unit vector along \vec{k} and



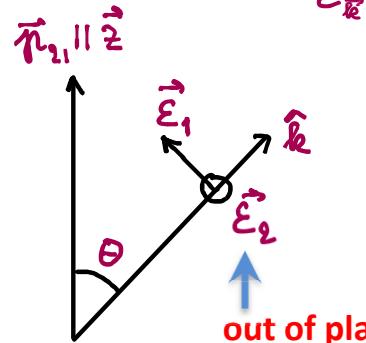
$$D(\omega_{\vec{k}}) = \frac{V}{(2\pi)^3} \frac{\omega_{\vec{k}}^2}{C^3}$$

mode density in shell of
 \vec{k} -space of radius $\omega_{\vec{k}}/c$

We define the “polarization average”

$$\overline{|\tilde{g}(\omega_{\vec{k}})|^2} = \sum_{\lambda} \int d(\hat{k}) |\tilde{g}_{\vec{k}, \lambda}|^2 = \frac{1}{k^2} \left(\frac{\hbar \omega_{\vec{k}}}{2 \epsilon_0 V} \right) \int d(\hat{k}) \sum_{\lambda} |\vec{p}_{21} \cdot \vec{\varepsilon}_{\vec{k}, \lambda}|^2$$

$\vec{p}_{21} \parallel \hat{k}$



in polar coordinates

$$\sum_{\lambda} |\vec{p}_{21} \cdot \vec{\varepsilon}_{\vec{k}, \lambda}|^2 = \sin^2 \theta |\vec{p}_{12}|^2$$

no φ dependence

$$\begin{aligned} \int d(\hat{k}) \sum_{\lambda} |\vec{p}_{21} \cdot \vec{\varepsilon}_{\vec{k}, \lambda}|^2 &= \int_0^{2\pi} d\phi \int_0^1 d(\cos \theta) \sin^2 \theta |\vec{p}_{12}|^2 \\ &= 2\pi |\vec{p}_{12}|^2 \int_{-1}^1 du (1-u^2) = \boxed{\frac{8\pi}{3} |\vec{p}_{12}|^2} \end{aligned}$$

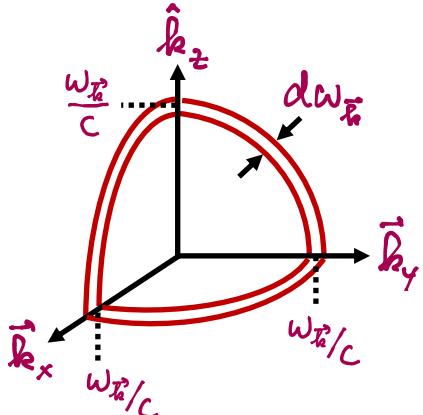
Wigner-Weisskopf Theory of Spontaneous Decay

$$\dot{c}_{2,0}(t) = - \sum_{\vec{k}, \lambda} |g_{\vec{k}, \lambda}|^2 \int_0^t e^{i(\omega_{\vec{k}} - \omega_{\vec{k}})(t-t')} c_{2,0}(t') dt'$$

Convert sum to integral over modes:

$$\begin{aligned} \sum_{\vec{k}} &\rightarrow \int d^3 \vec{k} \mathcal{D}(\vec{k}) = \int \hbar^2 d(\hat{\hbar}) d\vec{k} \mathcal{D}(\vec{k}) \\ &= \int d(\hat{\hbar}) d\omega_{\vec{k}} \frac{\omega_{\vec{k}}^2}{C^3} \mathcal{D}(\vec{k}) = \int d(\hat{\hbar}) d\omega_{\vec{k}} \mathcal{D}(\omega_{\vec{k}}) \end{aligned}$$

where $\hat{\hbar} = \vec{k}/\hbar$ is a unit vector along \vec{k} and



$$\mathcal{D}(\omega_{\vec{k}}) = \frac{V}{(2\pi)^3} \frac{\omega_{\vec{k}}^2}{C^3}$$

mode density in shell of
 \vec{k} -space of radius $\omega_{\vec{k}}/c$

Thus, in the Continuum Limit

$$\sum_{\vec{k}, \lambda} |g_{\vec{k}, \lambda}|^2 \rightarrow \int_0^\infty d\omega_{\vec{k}} \mathcal{D}(\omega_{\vec{k}}) \sum_{\lambda} \int d(\hat{\hbar}) |g_{\vec{k}, \lambda}|^2$$

We define the “polarization average”

$$\begin{aligned} \overline{|g(\omega_{\vec{k}})|^2} &= \sum_{\lambda} \int d(\hat{\hbar}) |g_{\vec{k}, \lambda}|^2 = \\ &= \frac{1}{\hbar^2} \left(\frac{\hbar \omega_{\vec{k}}}{2\epsilon_0 V} \right) \int d(\hat{\hbar}) \sum_{\lambda} |\vec{p}_{21} \cdot \vec{\varepsilon}_{\vec{k}, \lambda}|^2 \end{aligned}$$

$\vec{p}_{21} \parallel \hat{\vec{k}}$

$\vec{\varepsilon}_1$

$\vec{\varepsilon}_2$

out of plane

in polar coordinates

$\sum_{\lambda} |\vec{p}_{21} \cdot \vec{\varepsilon}_{\vec{k}, \lambda}|^2 = \sin^2 \theta |\vec{p}_{12}|^2$

no ϕ dependence

$$\begin{aligned} \int d(\hat{\hbar}) \sum_{\lambda} |\vec{p}_{21} \cdot \vec{\varepsilon}_{\vec{k}, \lambda}|^2 &= \int_0^{2\pi} d\phi \int_0^1 d(\cos \theta) \sin^2 \theta |\vec{p}_{12}|^2 \\ &= 2\pi |\vec{p}_{12}|^2 \int_{-1}^1 du (1-u^2) = \boxed{\frac{8\pi}{3} |\vec{p}_{12}|^2} \end{aligned}$$

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$$\dot{c}_{2,0}(t) = - \sum_{\vec{k}, \lambda} |g_{\vec{k}, \lambda}|^2 \int_0^t e^{i(\omega_{\vec{k}} - \omega_{\vec{k}})(t-t')} c_{2,0}(t') dt'$$

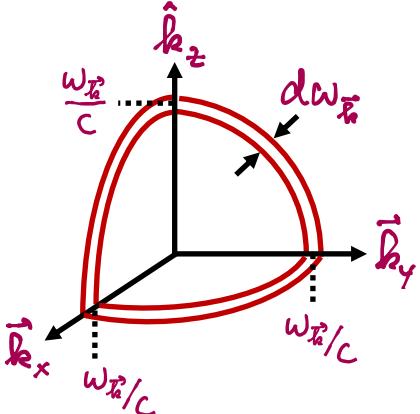
Thus, in the Continuum Limit

$$\sum_{\vec{k}, \lambda} |g_{\vec{k}, \lambda}|^2 \rightarrow \int_0^\infty d\omega_{\vec{k}} \mathcal{D}(\omega_{\vec{k}}) \sum_\lambda \int d(\hat{\vec{k}}) |g_{\vec{k}, \lambda}|^2$$

Convert sum to integral over modes:

$$\begin{aligned} \sum_{\vec{k}} &\rightarrow \int d^3 \vec{k} \mathcal{D}(\vec{k}) = \int \vec{k}^2 d(\hat{\vec{k}}) d\vec{k} \mathcal{D}(\vec{k}) \\ &= \int d(\hat{\vec{k}}) d\omega_{\vec{k}} \frac{\omega_{\vec{k}}^2}{C^3} \mathcal{D}(\vec{k}) = \int d(\hat{\vec{k}}) d\omega_{\vec{k}} \mathcal{D}(\omega_{\vec{k}}) \end{aligned}$$

where $\hat{\vec{k}} = \vec{k}/k$ is a unit vector along \vec{k} and



$$\mathcal{D}(\omega_{\vec{k}}) = \frac{V}{(2\pi)^3} \frac{\omega_{\vec{k}}^2}{C^3}$$

mode density in shell of
 \vec{k} -space of radius w_k/c

$$\begin{aligned} \overline{|g(\omega_{\vec{k}})|^2} &= \sum_\lambda \int d(\hat{\vec{k}}) |g_{\vec{k}, \lambda}|^2 = \\ &= \frac{1}{\hbar^2} \left(\frac{\hbar \omega_{\vec{k}}}{2\epsilon_0 V} \right) \int d(\hat{\vec{k}}) \sum_\lambda |\vec{p}_{2i} \cdot \vec{\varepsilon}_{\vec{k}, \lambda}|^2 \end{aligned}$$

$\vec{p}_{2i} \parallel \hat{\vec{k}}$

in polar coordinates

$$\sum_\lambda |\vec{p}_{2i} \cdot \vec{\varepsilon}_{\vec{k}, \lambda}|^2 = \sin^2 \theta |\vec{p}_{12}|^2$$

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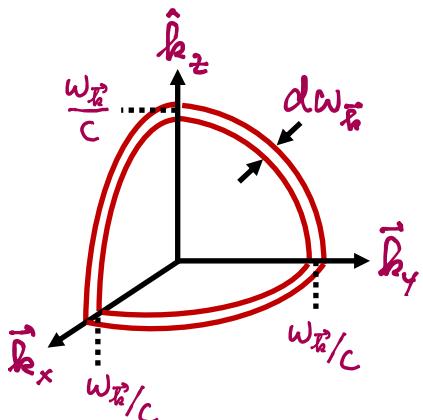
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$$\begin{aligned} \overline{|\mathbf{g}(\omega_{\vec{k}})|^2} &= \sum_\lambda \int d(\hat{\mathbf{k}}) |\mathbf{g}_{\vec{k}, \lambda}|^2 = \\ &= \frac{1}{\hbar^2} \left(\frac{\hbar \omega_{\vec{k}}}{2\epsilon_0 V} \right) \int d(\hat{\mathbf{k}}) \sum_\lambda |\vec{p}_{21} \cdot \vec{\varepsilon}_{\vec{k}, \lambda}|^2 \\ &= \frac{4\pi \omega_{\vec{k}}}{2\hbar \epsilon_0 V} |\vec{p}_{21}|^2 \end{aligned}$$

Putting it together:

$$\begin{aligned} \dot{c}_{2,0}(t) &= - \sum_{\vec{k}, \lambda} |\mathbf{g}_{\vec{k}, \lambda}|^2 \int_0^t e^{i(\omega_u - \omega_{\vec{k}})(t-t')} c_{2,0}(t') dt' \\ &= - \int_0^\infty d\omega_{\vec{k}} \overline{|\mathbf{g}(\omega_{\vec{k}})|^2} \mathcal{D}(\omega_{\vec{k}}) \int_0^t dt' e^{i(\omega_u - \omega_{\vec{k}})(t-t')} c_{2,0}(t') \end{aligned}$$

Wigner-Weisskopf Theory of Spontaneous Decay

Thus, in the Continuum Limit

$$\sum_{\vec{k}, \lambda} |g_{\vec{k}, \lambda}|^2 \rightarrow \int_0^\infty d\omega_{\vec{k}} D(\omega_{\vec{k}}) \sum_{\lambda} \int d(\hat{h}) |g_{\vec{k}, \lambda}|^2$$

We define the “polarization average”

$$\begin{aligned} \overline{|g(\omega_{\vec{k}})|^2} &= \sum_{\lambda} \int d(\hat{h}) |g_{\vec{k}, \lambda}|^2 = \\ &= \frac{1}{\hbar^2} \left(\frac{\hbar \omega_{\vec{k}}}{2 \epsilon_0 V} \right) \int d(\hat{h}) \sum_{\lambda} |\vec{p}_{21} \cdot \vec{\Sigma}_{\vec{k}, \lambda}|^2 \\ &= \frac{4 \pi \omega_{\vec{k}}}{2 \hbar \epsilon_0 V} |\vec{p}_{21}|^2 \end{aligned}$$

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$$\begin{aligned} \dot{c}_{2,0}(t) &= - \sum_{\vec{k}, \lambda} |g_{\vec{k}, \lambda}|^2 \int_0^t e^{i(\omega_{21} - \omega_{\vec{k}})(t-t')} c_{2,0}(t') dt' \\ &= - \int_0^\infty d\omega_{\vec{k}} \overline{|g(\omega_{\vec{k}})|^2} D(\omega_{\vec{k}}) \int_0^t dt' e^{i(\omega_{21} - \omega_{\vec{k}})(t-t')} c_{2,0}(t') \end{aligned}$$

Regular Perturbation Theory

Validity is limited to short times, such that the probability amplitude of the initial state does not change much. That allows us to approximate $c_{2,0}(t') \approx c_{2,0}(0)$. That does not work here.

Wigner-Weisskopf approximation:

The integral over time reduces to

$$\int_0^t dt' e^{i(\omega_{21} - \omega_{\vec{k}})(t-t')} \propto \delta(t-t')$$

on time scales $t-t' \gg |\omega_{21} - \omega_{\vec{k}}|^{-1}$

Thus, the integral that gives us $\dot{c}_{2,0}(t)$ is only sensitive to the value of $c_{2,0}(t')$ at times t' infinitesimally close to t . This allows us to replace $c_{2,0}(t')$ with $c_{2,0}(t)$ for any value of t .

Wigner-Weisskopf Theory of Spontaneous Decay

Thus, in the Continuum Limit

$$\sum_{\vec{k}, \lambda} |g_{\vec{k}, \lambda}|^2 \rightarrow \int_0^\infty d\omega_{\vec{k}} D(\omega_{\vec{k}}) \sum_{\lambda} \int d(\hat{h}) |g_{\vec{k}, \lambda}|^2$$

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Putting it together:

$$\begin{aligned} \dot{C}_{2,0}(t) &= - \sum_{\vec{k}, \lambda} |g_{\vec{k}, \lambda}|^2 \int_0^t e^{i(\omega_{21} - \omega_{\vec{k}})(t-t')} C_{2,0}(t') dt' \\ &= - \int_0^\infty d\omega_{\vec{k}} \overline{|g(\omega_{\vec{k}})|^2} D(\omega_{\vec{k}}) \int_0^t dt' e^{i(\omega_{21} - \omega_{\vec{k}})(t-t')} C_{2,0}(t') \end{aligned}$$

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$$\begin{aligned} \dot{C}_{2,0}(t) &= - \sum_{\vec{k}, \lambda} |g_{\vec{k}, \lambda}|^2 \int_0^t e^{i(\omega_{21} - \omega_{\vec{k}})(t-t')} C_{2,0}(t') dt' \\ &= - \int_0^\infty d\omega_{\vec{k}} \overline{|g(\omega_{\vec{k}})|^2} D(\omega_{\vec{k}}) \int_0^t dt' e^{i(\omega_{21} - \omega_{\vec{k}})(t-t')} C_{2,0}(t') \end{aligned}$$

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Wigner-Weisskopf approximation:

The integral over time reduces to

$$\int_0^t dt' e^{i(\omega_{21} - \omega_{\text{R}})(t-t')} \propto \delta(t-t')$$

on time scales $t-t' \gg |\omega_{21} - \omega_{\text{R}}|^{-1}$



$$\begin{aligned}\dot{C}_{2,0}(t) &= - \sum_{\text{R}, \lambda} |g_{\text{R}, \lambda}|^2 \int_0^t e^{i(\omega_{21} - \omega_{\text{R}})(t-t')} C_{2,0}(t') dt' \\ &= - \int_0^\infty d\omega_{\text{R}} |g(\omega_{\text{R}})|^2 \mathcal{D}(\omega_{\text{R}}) \int_0^t dt' e^{i(\omega_{21} - \omega_{\text{R}})(t-t')} C_{2,0}(t')\end{aligned}$$

Rest of Lecture: Work with this Equation

First: This eq. is of the form $\dot{C}_{2,0}(t) = \beta C_{2,0}(t)$



$\left\{ \begin{array}{l} \text{solutions oscillate at freq. } \text{Im}[\beta] \\ \text{and grow or decay at rate } \text{Re}[\beta] \end{array} \right.$

Next: Atoms couple weakly to the vacuum

$C_{2,0}(t)$ changes slowly on timescale ω_{21}^{-1} , evolving at a rate Γ to be determined.

That means we can let $t \rightarrow \infty$ on timescale ω_{21}^{-1} while still keeping $t \ll \Gamma^{-1}$

Defining $-i\zeta(\omega_{\text{R}} - \omega_{21}) = \int_0^\infty dt' e^{i(\omega_{21} - \omega_{\text{R}})(t-t')}$

We can then rewrite

$$\dot{C}_{2,0}(t) = - \int_0^\infty d\omega_{\text{R}} |g(\omega_{\text{R}})|^2 \mathcal{D}(\omega_{\text{R}}) [-i\zeta(\omega_{\text{R}} - \omega_{21})] C_{2,0}(t)$$

Wigner-Weisskopf Theory of Spontaneous Decay

Rest of Lecture: Work with this Equation

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We can then rewrite

$$\dot{C}_{2,0}(t) = - \int_0^\infty d\omega_{21} |g(\omega_{21})|^2 D(\omega_{21}) [-i\zeta(\omega_{21} - \omega_{21})] C_{2,0}(t)$$

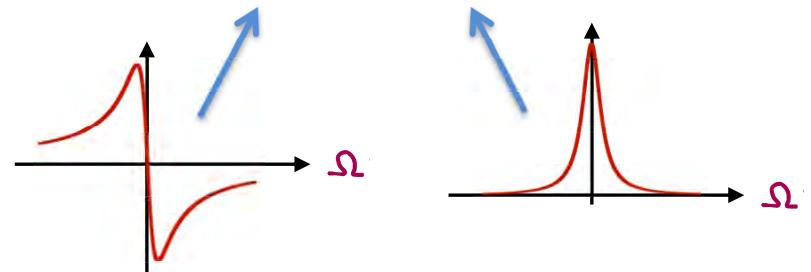
Math Details: Consider the function

$$\xi(\Omega) = i \int_0^\infty d\tau e^{-i\Omega\tau}, \quad \Omega = \omega_{21} - \omega_{21}$$

Approximate the integral as

$$\xi(\Omega) = \lim_{\varepsilon \rightarrow 0^+} i \int_0^\infty d\tau e^{-i\Omega\tau - \varepsilon\tau} = \lim_{\varepsilon \rightarrow 0^+} \left(\frac{i}{-i\Omega - \varepsilon} \right)$$

$$= \lim_{\varepsilon \rightarrow 0^+} \left(\frac{-\Omega^2}{\Omega^2 + \varepsilon^2} + i \frac{\varepsilon}{\Omega^2 + \varepsilon^2} \right)$$



$$= P\left(\frac{1}{\Omega}\right) + i\pi\delta(\Omega)$$

 Cauchy's Principal Part

Wigner-Weisskopf Theory of Spontaneous Decay

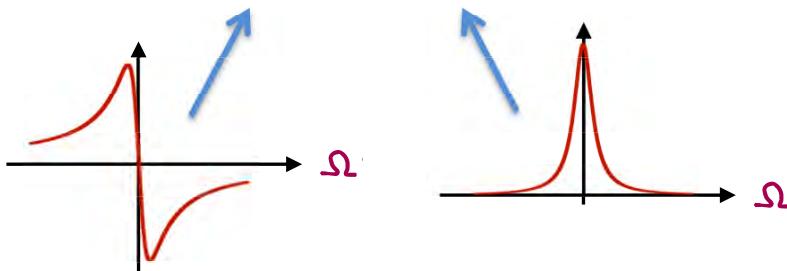
Math Details: Consider the function

$$\xi(\Omega) = i \int_0^\infty d\tau e^{-i\Omega\tau}, \quad \Omega = \omega_{\vec{k}} - \omega_{\vec{q}}$$

Approximate the integral as

$$\xi(\Omega) = \lim_{\varepsilon \rightarrow 0^+} i \int_0^\infty d\tau e^{-i\Omega\tau - \varepsilon\tau} = \lim_{\varepsilon \rightarrow 0^+} \left(\frac{i}{-i\Omega - \varepsilon} \right)$$

$$= \lim_{\varepsilon \rightarrow 0^+} \left(\frac{-\Omega^2}{\Omega^2 + \varepsilon^2} + i \frac{\varepsilon}{\Omega^2 + \varepsilon^2} \right)$$



$$= P\left(\frac{i}{\Omega}\right) + i\pi\delta(\Omega)$$



Cauchy's Principal Part

Thus,

$$\dot{c}_{2,0}(t) \approx - \left[\int_0^\infty d\omega_{\vec{k}} |g(\omega_{\vec{k}})|^2 D(\omega_{\vec{k}}) \left(\pi\delta(\omega_{21} - \omega_{\vec{k}}) - iP\left(\frac{1}{\omega_{21} - \omega_{\vec{k}}}\right) \right) \right] c_{2,0}(t)$$

which is of the form

$$\dot{c}_{2,0} = - \left(\frac{A_{21}}{2} - i\delta \right) c_{2,0}(t) \quad \rightarrow$$

$$c_{2,0}(t) = e^{-A_{21}/2 t} e^{i\delta(t)} c_{2,0}(0)$$

Decay Rate:

$$A_{21} = - \int_0^\infty d\omega_{\vec{k}} 2\pi |g(\omega_{\vec{k}})|^2 D(\omega_{\vec{k}}) \delta(\omega_{21} - \omega_{\vec{k}})$$

$$= 2\pi |g(\omega_{21})|^2 D(\omega_{21})$$

$$= 2\pi \times \frac{4\pi C_{21}}{3\hbar\epsilon_0 V} |\vec{p}_{21}|^2 \times \frac{V}{(2\pi)^3} \frac{\omega_{21}^2}{C^3}$$



Wigner-Weisskopf Theory of Spontaneous Decay

Thus,

$$\dot{C}_{2,0}(t) \approx - \left[\int_0^\infty d\omega_{\vec{k}} |g(\omega_{\vec{k}})|^2 D(\omega_{\vec{k}}) \left(\pi \delta(\omega_{21} - \omega_{\vec{k}}) - i \frac{1}{\omega_{21} - \omega_{\vec{k}}} \right) \right] C_{2,0}(t)$$

which is of the form

$$\begin{aligned} \dot{C}_{2,0} &= - \left(\frac{A_{21}}{2} - i\delta \right) C_{2,0}(t) \quad \rightarrow \\ C_{2,0}(t) &= e^{-A_{21}/2 t} e^{i\delta(t)} C_{2,0}(0) \end{aligned}$$

Decay Rate:

$$\begin{aligned} A_{21} &= - \int_0^\infty d\omega_{\vec{k}} 2\pi |g(\omega_{\vec{k}})|^2 D(\omega_{\vec{k}}) \delta(\omega_{21} - \omega_{\vec{k}}) \\ &= 2\pi |g(\omega_{21})|^2 D(\omega_{21}) \\ &= 2\pi \times \frac{4\pi c \omega_{21}}{3\hbar \epsilon_0 V} |\vec{p}_{21}|^2 \times \frac{V}{(2\pi)^3} \frac{\omega_{21}^2}{c^3} \end{aligned}$$

$$A_{21} = \frac{\omega_{21}^3 |\vec{p}_{21}|^2}{3\pi \epsilon_0 \hbar c^3} = \frac{8\pi^2 |\vec{p}_{21}|^2}{3\pi \hbar c^3}$$

This is the value used earlier!

Frequency shift:

$$\delta = \lim_{\varepsilon \rightarrow 0} \left(i \int_0^\infty d\omega_{\vec{k}} |\overline{g(\omega_{\vec{k}})}|^2 D(\omega_{\vec{k}}) \frac{\omega_{21} - \omega_{\vec{k}}}{(\omega_{21} - \omega_{\vec{k}})^2 + \varepsilon^2} \right)$$

Integral is not well behaved!

Further development by Willis Lamb



Lamb shift

Wigner-Weisskopf Theory of Spontaneous Decay

$$A_{21} = \frac{\omega_{21}^3 |\vec{p}_{21}|^2}{3\pi \epsilon_0 \hbar c^3} = \frac{8\pi^2 |\vec{p}_{21}|^2}{3\pi \hbar c^3}$$

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Frequency shift:

$$\delta = \lim_{\varepsilon \rightarrow 0} \left(i \int_0^\infty d\omega_{kk} [\overline{g(\omega_{kk})}]^2 D(\omega_{kk}) \frac{\omega_{21} - \omega_{kk}}{(\omega_{21} - \omega_{kk})^2 + \varepsilon^2} \right)$$

Integral is not well behaved!

Further development by Willis Lamb



Lamb shift

Hydrogen 2P lifetime:

- accurate calculations for

$$\lambda = 121.57 \text{ nm}$$

$$|\vec{p}_{21}| = 0.745 a_0 e = \frac{4\pi \epsilon_0 \hbar^2}{m_e e^2} \times 0.745 e$$



$$(A_{21})^{-1} = \tau_{2p} = 1.59 \text{ ns}$$

Measured: $\tau_{2p} = 1.60 \text{ ns}$