Quantum States of the Quantized Field

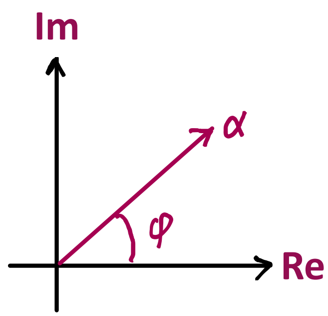
Amplitude and Phase

- Key characteristics of classical fields
- Need equivalents for quantum fields

Classical Field

$$E(z,t) = \mathcal{E}_k \boxed{\alpha} e^{-i(\omega t - kz)} + c.c$$

$|\alpha| e^{i\varphi}$



Quantum Field

$$\hat{E}(z,t) = \mathcal{E}_k \boxed{\hat{a}} e^{-i(\omega t - kz)} + H.C.$$

Non-Hermitian!
Separate in amplitude & phase?

Consider operators

$$\hat{a} = (\hat{N}+1)^{1/2} \hat{\exp}(i\varphi)$$
$$\hat{a}^{+} = \hat{\exp}(-i\varphi)(\hat{N}+1)^{1/2}$$

"phase" "amplitude"

$$\hat{\exp}(i\varphi) = (\hat{N}+1)^{-1/2}\hat{a}$$
$$\hat{\exp}(-i\varphi) = \hat{a}^{+}(\hat{N}+1)^{-1/2}$$

"Phase operators"

$$\hat{\exp}(i\varphi)\hat{\exp}(-i\varphi) = 1 \qquad \hat{\exp}(i\varphi) = \hat{\exp}(-i\varphi)^{+}$$
$$\hat{\exp}(-i\varphi)\hat{\exp}(i\varphi) = 1 \qquad = [\hat{\exp}(-i\varphi)]^{-1}$$

- Analogous to classical phases
- Non-Hermitian, NOT observables

Quadrature operators?

$$\hat{\cos}\varphi = \frac{1}{2}[\hat{\exp}(i\varphi) + \hat{\exp}(-i\varphi)]$$
$$= \frac{1}{2}[(\hat{N}+1)^{-1/2}\hat{a} + \hat{a}^{+}(\hat{N}+1)^{-1/2}]$$

$$\hat{\sin}\varphi = \frac{1}{2i}[\hat{\exp}(i\varphi) - \hat{\exp}(i\varphi)]$$
$$= \frac{1}{2i}[(\hat{N}+1)^{-1/2}\hat{a} - \hat{a}^{+}(\hat{N}+1)^{-1/2}]$$

- Hermitian -> observables
- but ultimately too cumbersome

Let's rewind and try again…

Quantum States of the Quantized Field

"Phase operators"

$$\hat{\exp}(i\varphi)\,\hat{\exp}(-i\varphi) = 1 \qquad \hat{\exp}(i\varphi) = \hat{\exp}(-i\varphi)^{+}$$

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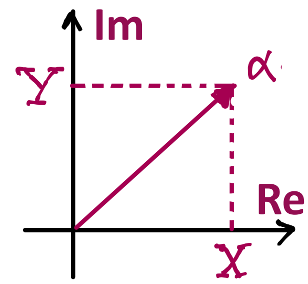
– Hermitian -> observables
– but ultimately too cumbersome

Let's rewind and try again…

Quadratures of the Classical Field – Take Two

$$E(z,t) = \underbrace{\mathcal{E}_k\,\alpha_k(t)}_{\text{complex amplitude for mode } e^{ikz}}\,e^{ikz} + c.c.$$



Define

$$\boxed{\begin{aligned} X(t) &= \mathrm{Re}\left[\alpha_k(t)\right] = \tfrac{1}{2}\left[\alpha_k(t) + \alpha_k^{*}(t)\right] = Q(t) \\ Y(t) &= \mathrm{Im}\left[\alpha_k(t)\right] = \tfrac{1}{2i}\left[\alpha_k(t) - \alpha_k^{*}(t)\right] = P(t) \end{aligned}}$$

Quantization: $\alpha \rightarrow \hat{a},\quad \alpha^{*} \rightarrow \hat{a}^{+}$

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– same info, easier to work with –

Quantum States of the Quantized Field

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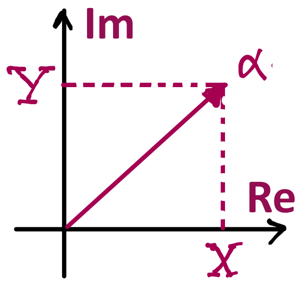

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Quantum States of the Field in Mode k

Number States (Foch states)

$$\boxed{\hat{a}^+\hat{a}|n\rangle = n|n\rangle}$$

$$\langle n|\hat{X}|n\rangle = \langle n|\hat{Y}|n\rangle = 0$$

$$\langle n|\hat{X}^2|n\rangle = \langle n|\hat{Y}^2|n\rangle = \tfrac{1}{2}(n + 1/2)$$

$$\boxed{\Delta X \Delta Y = \tfrac{1}{2}(n + 1/2)}$$

– HIGHLY non-classical, $\langle \hat{E} \rangle = 0$

– VERY hard to make for large n

Quantum States of the Quantized Field

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Coherent States (Quasi-classical states)

– Closest approximation to classical field

– See Cohen-Tannoudj, complement G_V

Definition: $|\psi\rangle$ is coherent (quasiclassical) iff

$$\langle\hat{X}(t)\rangle = \langle\psi|\hat{X}(t)|\psi\rangle = X(t),\quad \langle\hat{Y}(t)\rangle = Y(t)$$

$$\langle\hat{H}(t)\rangle = \hbar\omega\left(|\alpha(t)|^{2}+\tfrac{1}{2}\right)$$

noting that

$$\hat{X}(t) \propto \hat{a}(t) = \hat{a}(0)e^{-i\omega t}$$

$$\hat{Y}(t) \propto \hat{a}^{+}(t) = \hat{a}^{+}(0)e^{i\omega t}$$

equivalently

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Quantum States of the Quantized Field

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Cohen-Tannoudji, Lecture Notes

equivalently

Definition: a state $|\alpha\rangle$ is coherent iff

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

Finally, one can show

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

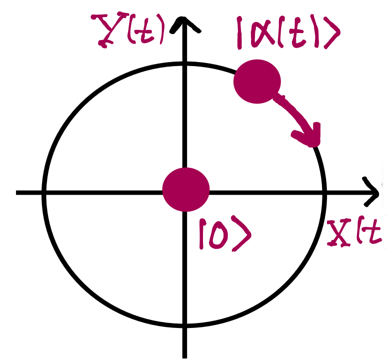
Physical properties

$$\langle \hat{X}(t)\rangle = \mathrm{Re}\left[\alpha(0)\, e^{-i\omega t}\right]$$

$$\langle \hat{Y}(t)\rangle = \mathrm{Im}\left[\alpha(0)\, e^{-i\omega t}\right]$$

$$\Delta X(t) = \Delta Y(t) = 1/2$$

$$\Delta X\, \Delta Y = 1/4$$



Quantum States of the Quantized Field

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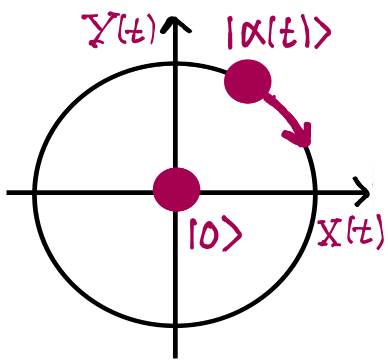


Photon statistics

Measure $\hat{N}$ ⇨ outcomes n

$$P(n) = \langle \alpha | n \rangle\langle n | \alpha \rangle = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$$

Poisson distribution w/

- mean $\bar{n} = |\alpha|^2$
- variance $\Delta n^2 = |\alpha|^2$

$$\Delta n = \sqrt{\bar{n}}$$ – Shot Noise

Quantum States of the Quantized Field

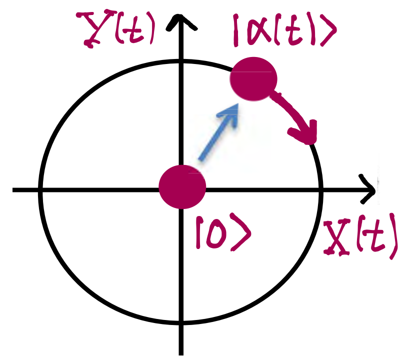
Photon statistics

Measure $\hat{N}$ ⇨ $\begin{cases} \text{outcomes } n \\ P(n) = \langle \alpha | n \rangle \langle n | \alpha \rangle = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} \end{cases}$

Poisson distribution w/ $\begin{cases} \text{mean} \quad \bar{n} = |\alpha|^2 \\ \text{variance} \quad \Delta n^2 = |\alpha|^2 \end{cases}$

$\boxed{\Delta n = \sqrt{\bar{n}}}$ **– Shot Noise**

More about Coherent States



Coherent States as translated Vacuum States?

Generating Coherent States from the Vacuum

<u>Definition:</u> $\hat{D}(\alpha) = e^{\alpha \hat{a}^{+} - \alpha^{*} \hat{a}}$

Unitary, equals translation

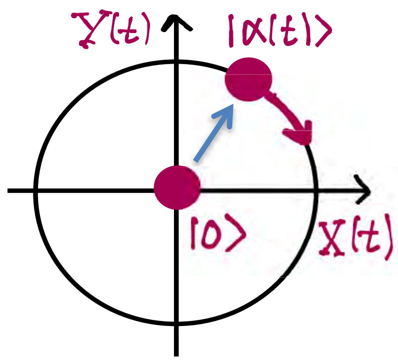
Glaubers formula **(from BCH formula)**

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{1}{2}[\hat{A},\hat{B}]}$$

for $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$

Quantum States of the Quantized Field

More about Coherent States



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Apply to $[\underbrace{\alpha\hat{a}^+}_{\hat{A}}, \underbrace{-\alpha^*\hat{a}}_{\hat{B}}] = \underbrace{\alpha^*\alpha}_{[\hat{A},\hat{B}]}$

$$\boxed{\hat{D}(\alpha) = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^+} e^{-\alpha^*\hat{a}}}$$

Remember: $\hat{a}|0\rangle = 0$

$$e^{-\alpha^*\hat{a}}|0\rangle = \sum_n \frac{(-\alpha^*\hat{a})^n}{n!}|0\rangle = |0\rangle$$

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$$= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle = |\alpha\rangle$$

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Quantum States of the Quantized Field

Apply to $[\alpha\hat{a}^+, -\alpha^*\hat{a}] = \alpha^*\alpha$

$\hat{A}$ $\quad$ $\hat{B}$ $\quad$ $[\hat{A}, \hat{B}]$

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OK – $\hat{D}(\alpha)$ generates $|\alpha\rangle$ from the vacuum!

Rewrite:

$$\alpha\hat{a}^+ - \alpha^*\hat{a} = (\alpha-\alpha^*)\hat{X} + i(\alpha+\alpha^*)\hat{Y}$$

$$= i2Y\hat{X} + i2X\hat{Y}$$

where $X = \langle\alpha|\hat{X}|\alpha\rangle$, $Y = \langle\alpha|\hat{Y}|\alpha\rangle$

$$\hat{X} = \frac{1}{2}[\hat{a} + \hat{a}^+] \quad , \quad \hat{Y} = \frac{1}{2i}[\hat{a} - \hat{a}^+]$$

$$X = \frac{1}{2}[\alpha + \alpha^*] \quad , \quad Y = \frac{1}{2i}[\alpha - \alpha^*]$$

Quantum States of the Quantized Field

Apply to $$[\underset{\hat{A}}{\alpha\hat{a}^{+}}, \underset{\hat{B}}{-\alpha^{*}\hat{a}}] = \underset{[\hat{A},\hat{B}]}{\alpha^{*}\alpha}$$

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Glaubers formula again:

$$\boxed{\hat{D}(\alpha) = e^{i2Y\hat{X} + i2X\hat{Y}} = e^{-XY/4}\, e^{i2Y\hat{X}}\, e^{i2X\hat{Y}}}$$

Recall: $\hat{S}(q) = e^{-iq\hat{p}/\hbar}$ **translation by** q

$\hat{S}(p) = e^{-ip\hat{q}/\hbar}$ **translation by** p

where $q = q_0 X$, $p = p_0 Y$

$\hat{q} = q_0\hat{X}$, $\hat{p} = p_0\hat{Y}$ & $q_0 p_0 = 2\hbar$

Quantum States of the Quantized Field

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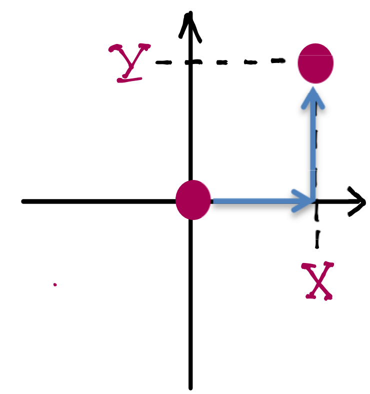
$\hat{q} = q_0\hat{X}$, $\hat{p} = p_0\hat{Y}$ & $q_0 p_0 = 2\hbar$

This gives us

$$\hat{S}(q) = \hat{S}(X) = e^{i2X\hat{Y}}, \quad \hat{S}(p) = \hat{S}(Y) = e^{i2Y\hat{X}}$$

$\hat{D}(\alpha)$ **translates along X then Y**



Quantum States of the Quantized Field

Coherent States from Classical Dipole Radiation

Classical Dipole $d(t) = d_0 \cos(\omega t)$ @ $t=0$

Quantized Field $\hat{E}(z) = \mathcal{E}_k (\hat{a} + \hat{a}^+)$

Dipole-Field Interaction

$$\hat{H} = \hbar\omega(\hat{a}^+\hat{a} + 1/2) + \hbar\lambda(t)(\hat{a}+\hat{a}^+)$$

$$\lambda(t) = -\frac{d(t)\mathcal{E}_k}{\hbar} = \lambda_0 \cos(\omega t)$$

Drive from $t=0$ to T

Homework Problem (voluntary)

$$\alpha(T) = -i\frac{\lambda_0}{2} e^{-i(\omega-\omega')T/2} \frac{\sin[(\omega-\omega')T/2]}{(\omega-\omega')/2}$$

Recall from Semi-Classical Laser Theory

$\langle \hat{p}(t) \rangle$ **drives** $\hat{E}(t)$

classical dipole + quantum fluctuations

coherent state + quantum fluctuations

For $t > T$ we have a coherent state

$$\alpha(t) = \alpha(T) e^{-i\omega(t-T)}$$

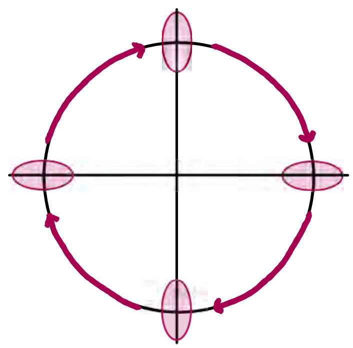
Quantum States of the Quantized Field

Squeezed States

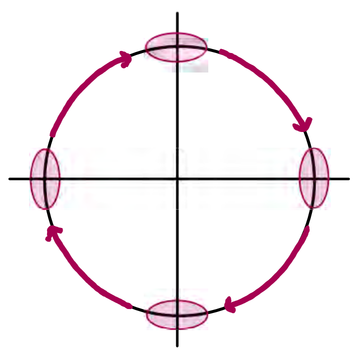
Minimum uncertainty states w/assymmetry

$$\Delta X \Delta Y = 1/4, \quad \Delta X(t) \neq \Delta Y(t)$$

Phase Squeezing



Amplitude Squeezing



Requires interaction with Nonlinear medium

Odds and Ends – Thermal States

$$Z = \mathrm{Tr}\left[e^{-\hat{H}/k_B T}\right]$$

$$\hat{\rho} = \sum_n P(n)|n\rangle\langle n| = \frac{1}{Z}\sum_n e^{-E_n/k_B T}|n\rangle\langle n|$$

$$= (1-q)\sum_n q^n |n\rangle\langle n|, \quad q = e^{-\hbar\omega/k_B T}$$

Mean Photon Number:

$$\bar{n} = \mathrm{Tr}(\hat{\rho}\hat{N}) = \sum_{n',n}\langle n'|(1-q)q^n|n\rangle\langle n|\hat{N}|n'\rangle$$

$$= (1-q)\sum_n n q^n = \frac{q}{1-q}$$

Photon Number Uncertainty:

$$\langle \hat{N}^2\rangle = (1-q)\sum_n n^2 q^n = \frac{q^2+q}{(1-q)}$$

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$$\Delta n^2 = \langle \hat{N}^2\rangle - \langle \hat{N}\rangle^2$$

$$= \frac{q^2+q}{(1-q)} - \frac{q^2}{(1-q)^2} = \frac{q}{(1-q)^2}$$

$$\bar{n} = \frac{q}{1-q}$$

Coherent State limit

$$\Delta n = \frac{\sqrt{q}}{1-q} = \sqrt{\bar{n}(\bar{n}+1)} \geq \sqrt{\bar{n}}$$

Optical Frequencies, Room Temperature:

$$\lambda = 1\,\mu m, \qquad T = 300\,K$$

$$q = 6.5\times 10^{-6}, \quad \bar{n} \sim 10^{-6}$$

Quantum States of the Quantized Field

Odds and Ends – Quantum-Classical Correspondence

Define a Translation Operator

$$\hat{T}_\alpha(t) = e^{\alpha^* e^{i\omega t}\hat{a} - \alpha e^{-i\omega t}\hat{a}^+} = \hat{D}(-\alpha e^{-i\omega t})$$

Use $[\hat{a}, \hat{F}(\hat{a}^+)] = dF(\hat{a}^+)/d\hat{a}^+$ **to show**

$$[\hat{a}, \hat{T}_\alpha] = \hat{a}\hat{T}_\alpha - \hat{T}_\alpha\hat{a} = -\alpha e^{-i\omega t}\hat{T}_\alpha$$

$$\Rightarrow \hat{T}_\alpha\hat{a} = \hat{a}\hat{T}_\alpha + \alpha e^{-i\omega t}\hat{T}_\alpha$$

$$\Rightarrow \hat{T}_\alpha\hat{a}\hat{T}_\alpha^+ = \hat{a} + \alpha e^{-i\omega t}$$

From this we get

$$\hat{E}_\perp' = \hat{T}_\alpha \hat{E}_\perp \hat{T}_\alpha^+ = \hat{T}_\alpha\left(\mathcal{E}_k \hat{a} e^{i\vec{k}\cdot\vec{r}} + H.C.\right)\hat{T}_\alpha^+$$

$$= \mathcal{E}_k \hat{a} e^{i\vec{k}\cdot\vec{r}} + H.C. + \mathcal{E}_k \alpha e^{-i(\omega t - \vec{k}\cdot\vec{r})} + C.C.$$

$$= \hat{E}_\perp + E_\perp^{Cl}(\alpha, t)$$

We also have $|\psi'(t)\rangle = \hat{T}_\alpha |\alpha(t)\rangle = |0\rangle$

Action of the unitary transformation $\hat{T}_\alpha(t)$

$$\boxed{\begin{aligned} &\hat{E}_\perp' = \hat{T}_\alpha(t)\hat{E}_\perp \hat{T}_\alpha(t)^+ = \hat{E}_\perp + E_\perp^{Cl}(\alpha, t) \\ &|\psi'(t)\rangle = T_\alpha(t)\,|\alpha(t)\rangle = |0\rangle \end{aligned}}$$

$\Downarrow$

We can work with

$\hat{E}_\perp, |\alpha(t)\rangle$ **or** $\hat{E}_\perp + E_\perp^{Cl}(\alpha, t), |0\rangle$

Validates Semiclassical Optics for strong Coherent Fields!